OUTLINING PROOFS IN CALCULUS

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Logical Deduction ... is the one and only true powerhouse of mathematical thinking.

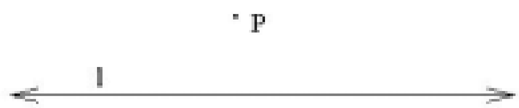
Jean Dieudonne

Conjecturing and demonstrating the logical validity of conjectures are the essence of the creative act of doing mathematics.

National Council of Teachers of Mathematics

Introduction

Consider a point p and a line l in some plane, with p not on l:



How many lines are there in the plane that pass through point p and that are parallel to line l? It seems clear, by what we mean by “point”, “line”, and “plane”, that there is just one such line. This assertion is logically equivalent to Euclid's 5th, or parallel, postulate (in the context of his other postulates).

In fact, this was seen as so obvious by everyone, mathematicians included, that for two thousand years mathematicians attempted to prove it. After all, if it was true, then it should be provable. One approach, taken independently by Carl F. Gauss, Janos Bolyai, and Nikolai Lobachevsky, was to assume the negation of the “obvious truth” and attempt to arrive at a logical contradiction. But they did not arrive at a contradiction. Instead, the logical consequences went on and on. They included “theorems”, all provable from the assumptions. In fact, they discovered what Bolyai called a “strange new universe”, and what we today call non-Euclidean geometry.

The strange thing is that the real, physical universe turns out to be non-Euclidean. Einstein's special theory of relativity uses a geometry developed by H. Minkowski, and his general theory, a geometry of Gauss and G. F. B. Riemann.[1]

Mathematics is about an imaginative universe—a world of ideas, but the imagination is constrained by logic. The basic idea behind proof in mathematics is that everything is exactly what its definition says it is. A proof that something has a property is a demonstration that the property follows logically from the definition alone. On the intuitive level, definitions serve to lead our imagination. In a formal proof, however, we are not allowed to use attributes of our imaginative ideas that don't follow logically solely from definitions and axioms relating undefined terms. This view of proof, articulated by David Hilbert, is accepted today by the mathematical community, and is the basis for research mathematics and graduate and upper-level undergraduate mathematics courses.

There is also a very satisfying aspect of “proof” that comes from our ability to picture situations—and to draw inferences from the pictures. In calculus, many, but not all, theorems have satisfactory picture proofs. Picture proofs are satisfying, because they enable us to see[2] the truth of the theorem. Rigorous proof in the sense of Hilbert has an advantage, not shared by picture proofs, that proof outlines are suggested by the very language in which theorems are expressed. Thus both picture proofs and rigorous proofs have advantages. Calculus is best seen using both types of proof.

A picture is an *example* of a situation covered by a theorem. The theorem, of course, is not true about the actual picture—the molecules of ink stain on the molecules of paper. It is true about the idealization that we intuit from the picture. When we see that the proof of a theorem follows from a picture, we see that the picture is in some sense completely general—that we can't draw another picture for which the theorem is false. To those mathematicians that are satisfied only by rigorous mathematics, such a situation would merely represent proof by lack of imagination: we

[1] See Marvin Greenberg, *Euclidean and Non-Euclidean Geometries*, W. H. Freeman, San Fransisco, 1980.
[2] The word “theorem” literally means “object of a vision”.

can't imagine any situation essentially different from the situation represented by the picture, and we conclude that because we can't imagine it that it doesn't exist. Non-Euclidean geometry is an example of a situation where possibilities not imagined are, nevertheless, not logically excluded. Gauss[3] kept his investigations in this area secret for years, because he wanted to avoid controversy, and because he thought that "the Boeotians"[4] would not understand. It is surely a profound thing that the universe, while it may not be picturable to us, is nevertheless logical, and that following the logical but unpicturable has unlocked deep truths about the universe. In mathematics, unpicturable but logical results are sometimes called *counterintuitive*.

Rigorous and picture proofs are *both* necessary to a good course in calculus and *both* within grasp[5]. To insist on a rigorous proof, where a picture has made everything transparent, *is* deadening. Mathematicians with refined intuition know that they could, if pressed, supply such a proof—and it therefore becomes unnecessary to actually do it. We therefore focus the method of outlining proofs on those theorems for which there is no satisfactory picture proof.

Our purpose is not to cover all theorems of calculus, but to do enough to enable a student to "catch on" to the method.[6] A detailed formal exposition of the method can be found in *Introduction to Proof in Abstract Mathematics* and *Deductive Mathematics—an introduction to proof and discovery for mathematicsd education.*

In calculus texts, "examples" are given that illustrate computational techniques, the use of certain ideas, or the solution of certain problems. In these notes, we give "examples" that illustrate the basic features of the method of discovering a proof outline.

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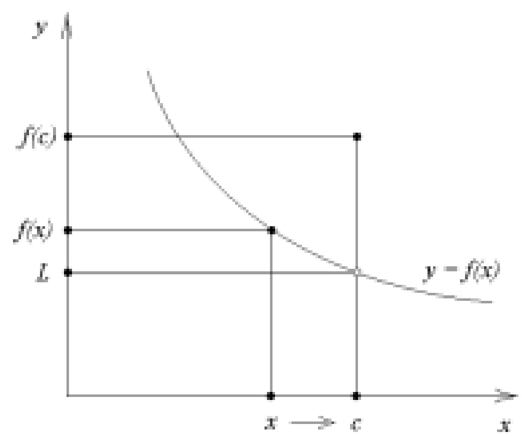
[3] One logical consequence of there being more than one line through p parallel to l is that the sum of the number of degrees in the angles of a triangle is less than 180°. Gauss made measurements of the angles formed by three prominent points, but his measurements were inconclusive. We know today that the difference between the Euclidean 180° and that predicted for his triangle by relativity would be too small to be picked up by his instruments.
[4] A term of derision. Today's "Boeotians" have derisive terms of their own—claiming courses that depend on rigorous proof have "rigor-mortis".
[5] The fundamentals of discovering proof outlines can be picked up in a relatively very short time—compared to the *years* of study of descriptive mathematics prerequisite to calculus.
[6] In the *American Math. Monthly* 102, May 1995, page 401, Charles Wells states: "A colleague of mine in computer science who majored in mathematics as an undergraduate has described how as a student he suddenly caught on that he could do at least B work in most math courses by merely rewriting the definitions of the terms involved in the questions and making a few … deductions." The reason this worked for Prof. Well's colleague is that definitions are basic for deductive mathematics—and upper-level math courses are taught from a deductive perspective. Our method for discovering proof steps is merely a systematic way of making the deductions—a way that can be taught.

Limits

Consider the limit of a function f—as the independent variable x approaches some value c. If f has such a limit, it is denoted by $\lim_{x \to c} f(x)$. This is read "the limit of f of x as x approaches c". Here is a picture of a function and its graph that expresses a situation where the limit is equal to a number L:



We see from the picture that as x "goes to", or "approaches" c, that, in turn, $f(x)$ "goes to", or "approaches" L. This is the intuitive idea of the limit of a function. It is necessary to capture this idea in a formal definition. Theorems and definitions in formal mathematics are expressed in a language that employs only 7 different types of sentences[7]. The types could be listed as:

for all (or for every)
there exists
if, then
or
and
not
iff

The formal definition of the limit of a function uses sentences of these types. Suppose that $\mathbb{R}$ denotes the set of real numbers and D is a subset of $\mathbb{R}$ on which a function f is defined. We have:

Definition For a function $f\colon D \to \mathbb{R}$ and real numbers c and L, define $\lim_{x \to c} f(x) = L$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x \in \mathbb{R}$, if $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$.

[7] —in addition to the sentences expressing relationships between the objects we are studying—such as numerical equality or set membership.

In a definition, we make the thing defined *logically equivalent* to the defining condition. Thus, "$\lim_{x\to c} f(x) = L$" is logically equivalent to its defining condition "for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x \in \mathbb{R}$, if $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$". Logical equivalence is generally denoted by the words "if and only if"—abbreviated "iff". Thus the wording of the definition should technically have been " $\lim_{x\to c} f(x) = L$ if and only if for every $\epsilon > 0$, there exists $\ldots$". In a definition however, the word "if", when used to relate the thing defined to its defining condition, always means "iff".

At this point it is imperative that you see that the intuitive idea of limit (conveyed by the picture and the concept of "movement") is captured exactly by the formal definition; that is, the formal condition (which begins with the words "for every $\epsilon > 0$") is true in situations where the limit exists according to our intuition, and the limit exists according to our intuition when the formal definition is true. You can't *prove* this connection between the intuitive idea and the formal definition. That is not within the realm of proof. But you must "see" the connection.

The whole idea behind formal proof in mathematics is that anything that is *proved* true about the limit of a function must be shown to be a consequence of the formal defining condition. The intuitive idea of limit should inform all we find to be true about limits. It should motivate our investigations. It can be used to convince ourselves or our sympathetic listeners of the truth of our investigations. But it cannot be used in a formal proof. Right in the middle of a formal proof we can't appeal, for example, to the physical idea of "motion".

A formal proof depends on the precise meaning (given by accepted rules of inference) of sentences of the seven types listed above. Theorems are expressed in terms of this formal language—as are definitions—and a theorem about the limit of a function, for example, must be shown to be true (in a formal proof) solely as a logical consequence of the definition of limit.

As we said in the introduction, what might appear to be a burden—the need to be formal—actually turns out to be a help in learning to do one's own proofs: the language in which formal mathematics is expressed provides the key to the structure of a proof. Here is the first example that illustrates the method of developing an outline of a proof structure.

Example 1 If $f(x) = 3x + 2$, then $\lim_{x\to 5} f(x) = 17$.

Most mathematical propositions or theorems assert that something is true (the *conclusion*) under certain conditions (the *hypotheses*). The first step in developing a proof of such a theorem or proposition is to identify, in a sentence or two, the hypotheses and conclusion. We will develop proofs themselves in a shaded area that will differentiate proofs from standard text—where we will talk *about* proofs. The first entry in the shaded area identifies the hypotheses and conclusion: it tells the reader what you are assuming and what you will show.

Proof:

Assume $f(x) = 3x + 2$. We will show $\lim_{x\to 5} f(x) = 17$.

The statement $f(x) = 3x + 2$ serves to define the function f. Function definitions are accomplished by giving a rule for what the function does to any element in its domain—some subset of the real numbers. The statement $f(x) = 3x + 2$ is really a customary abbreviation for the more complete, formal statement "for all $x \in \mathbb{R}$, $f(x) = 3x + 2$". "For all $x \in \mathbb{R}$" is read "for all x in $\mathbb{R}$". Function definitions always need to specify what the function does to each element in its domain. In general, we write $g\colon D \to \mathbb{R}$ to express the fact that the function g has domain D (a subset of $\mathbb{R}$) and codomain $\mathbb{R}$. For our function f here, $D = \mathbb{R}$. The phrase "for all $x \in D$" is always implied—even when it's not explicitly written in a function definition.

In our development, we will always take D to be either the entire set of real numbers or some open interval. This will keep proofs (at our introductory level) free from complicating special cases (such as one-sided limits), while nonetheless exhibiting the generality of the method.

The next step in developing the proof is to write the conclusion as the last statement—at the bottom of the proof. A proof is a sequence of statements that lead to the conclusion, so the last statement is always the conclusion. This leaves a gap in the proof[8], namely, all the statements needed to establish the conclusion. The method of outlining a proof will, in general, fill in many, if not all, of these statements.

Proof:

Assume $f(x) = 3x + 2$. We will show $\lim_{x \to 5} f(x) = 17$.

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$\lim_{x \to 5} f(x) = 17$.

The next step is to consider what it means for the conclusion $\lim_{x \to 5} f(x) = 17$ to be true. In a formal proof, this meaning can come only from the definition—in this case, the definition of limit. Here again is the definition:

Definition For a function $f \colon D \to \mathbb{R}$ and real numbers c and L, define $\lim_{x \to c} f(x) = L$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x \in \mathbb{R}$, if $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$.

The next step in the proof outlining process is to write what $\lim_{x \to 5} f(x) = 17$ means by definition, as the next-to-last statement in the proof. In a definition, the thing defined is logically equivalent to the defining condition. Thus, "$\lim_{x \to 5} f(x) = 17$" is logically equivalent to its defining condition "for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x \in \mathbb{R}$, if $0 < |x - 5| < \delta$, then $|f(x) - 17| < \epsilon$".

Proof:

Assume $f(x) = 3x + 2$. We will show $\lim_{x \to 5} f(x) = 17$.

.

.

.

For every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x \in \mathbb{R}$, if $0 < |x - 5| < \delta$, then $|f(x) - 17| < \epsilon$.

$\lim_{x \to 5} f(x) = 17$.

In order to prove $\lim_{x \to 5} f(x) = 17$, we prove (in the preceding statement) the defining condition for $\lim_{x \to 5} f(x) = 17$. This is the condition given in the definition applied to the function and limit in Example 1: $L = 17$, $D = \mathbb{R}$, and $c = 5$. There is no other choice, at this stage, since we have no theorems about limits. Once we have proved the defining condition, the conclusion will follow logically from it (by definition) since for any mathematical statement, we can substitute a logically equivalent one. Our task is now to prove the statement beginning with "for every $\epsilon > 0$, …". this has the form "for every $\epsilon > 0$, (lower-level statement involving ϵ)".

[8] At this point the gap is the entire proof.

Mathematics, not just calculus, but all areas of mathematics, can be formalized in terms of the 7 types of statements:

for all
there exists
if, then
or
and
not
iff

The method of proof outlining depends on rules for proving and using statements of the first 4 types. *For-all* statements have the form "for all $x \in A$, (lower-level statement involving x)", where A is some set. In particular, A can be the set $\mathbb{R}^+$ of positive real numbers. The statement "$x \in \mathbb{R}^+$" means exactly the same as the statement "$x > 0$", so that "for all $x \in \mathbb{R}^+$" means the same as "for all $x > 0$". This, in turn, means the same as "for every $x \in \mathbb{R}^+$" or "for every $x > 0$", since, if a statement is true for all elements in a set, then it is true for every one of them—and *vice versa*.

The defining condition for limit is therefore one form of a *for-all* statement. Its form "for every $\epsilon > 0$, (lower-level statement involving ϵ)" is equivalent to "for all $\epsilon > 0$, (lower-level statement involving ϵ)". We use calligraphic letters to represent statements. Thus $\mathcal{P}$ and $\mathcal{Q}$ may denote statements. We may write $\mathcal{P}(x)$, for example, if we wish to emphasize that the statement $\mathcal{P}$ involves the symbol x. Thus, a typical *for-all* statement is of the form "for all $x \in A$, $\mathcal{P}(x)$", where A is some set and $\mathcal{P}(x)$ is a statement involving x. The top-level *for-all* statement contains the lower-level statement $\mathcal{P}(x)$.

The way that mathematicians prove *for-all* statements is somewhat subtle. In order to prove a statement of the form "for all $x \in A$, $\mathcal{P}(x)$", select some element, say y, from the set A, and then show $\mathcal{P}(y)$ is true. The idea is to prove $\mathcal{P}(y)$ with no information about y, except that it is in A. Such an element of A is called an *arbitrary* or *given* element of A. If $\mathcal{P}(y)$ is true for an arbitrary element y of A, then it is true for all elements of A. The statement "for all $x \in A$, $\mathcal{P}(x)$" is therefore proved by this procedure.

In order to prove the statement "for all $x \in A$, $\mathcal{P}(x)$" that appears at the bottom of a gap in a proof, we insert two new statements in the gap. First, write "let x be arbitrarily chosen in A", "let $x \in A$ be arbitrary", "let $x \in A$ be given", or some such phrase at the top of the gap. This defines the symbol x. Second, write $\mathcal{P}(x)$ at the bottom of the gap. It is then necessary to prove $\mathcal{P}(x)$—to fill the remaining gap. In our developing proof, we have inserted "let $\epsilon > 0$ be given" at the top of the gap, and the statement "there exists $\delta > 0$ such that $\ldots$" at the bottom:

Proof:

Assume $f(x) = 3x + 2$. We will show $\lim_{x \to 5} f(x) = 17$.

Let $\epsilon > 0$ be given.

.

.

.

There exists $\delta > 0$ such that for all $x \in \mathbb{R}$, if $0 < |x - 5| < \delta$, then $|f(x) - 17| < \epsilon$.

For every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x \in \mathbb{R}$, if $0 < |x - 5| < \delta$, then $|f(x) - 17| < \epsilon$.

$\lim_{x \to 5} f(x) = 17$.

Our task is now to prove the *new* statement at the bottom of the gap, which has the form "there exists $\delta > 0$ such that (lower-level statement involving δ)". In order to prove such statements (called *there-exists* statements) it is necessary for us to define δ, to show $\delta > 0$ (if this is not obvious), and to show the lower-level statement involving δ. Thus a statement that begins "let $\delta = \ldots$" needs to be inserted somewhere in the gap (we'll leave space above and below the definition of δ)[9], and the lower-level piece of the *there-exists* statement needs to be put at the bottom of the gap:

Proof:

Assume $f(x) = 3x + 2$. We will show $\lim_{x \to 5} f(x) = 17$.

Let $\epsilon > 0$ be given.

.

Let $\delta = \ldots$

.

.

For all $x \in \mathbb{R}$, if $0 < |x - 5| < \delta$, then $|f(x) - 17| < \epsilon$

There exists $\delta > 0$ such that for all $x \in \mathbb{R}$, if $0 < |x - 5| < \delta$, then $|f(x) - 17| < \epsilon$.

For every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x \in \mathbb{R}$, if $0 < |x - 5| < \delta$, then $|f(x) - 17| < \epsilon$.

$\lim_{x \to 5} f(x) = 17$.

The statement "let $\delta = \ldots$" is like an assignment statement in computer science. It will be necessary, as the proof develops, for us to define δ in terms of symbols that already exist in the proof (in the hypotheses or preceding statements). At this stage we don't know how to define δ, but we know where we will need to do it. So we rough in the words "let $\delta = \ldots$".

Our task is now to prove the statement "for all $x \in \mathbb{R}$, if $0 < |x - 1| < \delta$, then $|f(x) - 2| < \epsilon$". It is a *for-all* statement, at the top level. Our rule for proving such statements dictates that we insert a line such as "let $x \in \mathbb{R}$ be arbitrary" at the top of the gap, and the line "if $0 < |x - 1| < \delta$, then $|f(x) - 2| < \epsilon$" at the bottom:

Proof:

Assume $f(x) = 3x + 2$. We will show $\lim_{x \to 5} f(x) = 17$.

Let $\epsilon > 0$ be given.

.

Let $\delta = \ldots$

Let $x \in \mathbb{R}$ be arbitrary.

.

.

If $0 < |x - 5| < \delta$, then $|f(x) - 17| < \epsilon$.

For all $x \in \mathbb{R}$, if $0 < |x - 5| < \delta$, then $|f(x) - 17| < \epsilon$.

There exists $\delta > 0$ such that for all $x \in \mathbb{R}$, if $0 < |x - 5| < \delta$, then $|f(x) - 17| < \epsilon$.

For every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x \in \mathbb{R}$, if $0 < |x - 5| < \delta$, then $|f(x) - 17| < \epsilon$.

$\lim_{x \to 5} f(x) = 17$.

The next step is to prove the new statement at the bottom of the gap—which is of the form *if* $\mathcal{P}$*, then* $\mathcal{Q}$—and is called an *if-then* statement or *implication*. In order to prove statements of this

[9] The proof outlining method depends upon inserting statements that are dictated by the form of the statement that is at the bottom of the gap. For the purposes of this method, the new gap will be *below* the statement "let $\delta = \ldots$". The space *above* this statement will be merely set aside for any work needed to define δ (say, by using the hypotheses). Although "let $\delta = \ldots$" appears at the bottom of this space, it does not dictate the insertion of other statements.

form, assume that $\mathcal{P}$ is true (at the top of the gap) and show that $\mathcal{Q}$ is true under this assumption (at the bottom of the gap). In our proof we have:

Proof:

Assume $f(x) = 3x + 2$. We will show $\lim\limits_{x \to 5} f(x) = 17$.

Let $\epsilon > 0$ be given.

.

.

Let $\delta = \dots$

Let $x \in \mathbb{R}$ be arbitrary.

Assume $0 < |x - 5| < \delta$.

.

.

$|f(x) - 17| < \epsilon$

If $0 < |x - 5| < \delta$, then $|f(x) - 17| < \epsilon$.

For all $x \in \mathbb{R}$, if $0 < |x - 5| < \delta$, then $|f(x) - 17| < \epsilon$.

There exists $\delta > 0$ such that for all $x \in \mathbb{R}$, if $0 < |x - 5| < \delta$, then $|f(x) - 17| < \epsilon$.

For every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x \in \mathbb{R}$, if $0 < |x - 5| < \delta$, then $|f(x) - 17| < \epsilon$.

$\lim\limits_{x \to 5} f(x) = 17$.

Proof outlining is done by working up from the bottom, beginning with the conclusion. At each stage, statements are added to the outline as they are dictated by the top-level form of the statement at the bottom of the gap—what it means for such a statement to be true. This may be by definition, or because the statement has a certain logical form (*there exists*, *for all*, and so on). In the latter case, an inference rule (such as those on page 8) will dictate statements to be put in the proof gap. When the new statement at the bottom of the gap is not any one of the logical forms at the top level, and has not been defined, we can proceed with this program no further, When this stage is reached (and not before), it is time to use the hypotheses to help bridge the remaining gap. We are now at this stage in our developing proof. At the top level, the statement at the bottom of the gap, $|f(x) - 17| < \epsilon$, is of the form *number* $<$ *number*—which has not been defined in our development. Thus we use the hypothesis (which defines the function f). We do this by substituting $3x + 2$ for $f(x)$—as we work up from the bottom.

Proof:

Assume $f(x) = 3x + 2$. We will show $\lim\limits_{x \to 5} f(x) = 17$.

Let $\epsilon > 0$ be given.

.

.

Let $\delta = \dots$

Let $x \in \mathbb{R}$ be arbitrary.

Assume $0 < |x - 5| < \delta$

.

$|(3x + 2) - 17| < \epsilon$

$|f(x) - 17| < \epsilon$ (by hypothesis and the definition of f)

If $0 < |x - 5| < \delta$, then $|f(x) - 17| < \epsilon$.

For all $x \in \mathbb{R}$, if $0 < |x - 5| < \delta$, then $|f(x) - 17| < \epsilon$.

There exists $\delta > 0$ such that for all $x \in \mathbb{R}$, if $0 < |x - 5| < \delta$, then $|f(x) - 17| < \epsilon$.

For every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x \in \mathbb{R}$, if $0 < |x-5| < \delta$, then $|f(x) - 17| < \epsilon$.

$\lim_{x \to 5} f(x) = 17$.

In a proof, one should always note where the hypotheses are used, so we have noted in parentheses that $|f(x) - 17| < \epsilon$ follows from $|(3x+2) - 17| < \epsilon$ by hypothesis and the definition of f.

This brings us to the end of the proof outlining process. The remaining tasks have been identified: we need to show that $|(3x+2) - 17| < \epsilon$ follows from $0 < |x-5| < \delta$, and we are allowed to define δ any way we wish, in order to do this. It seems clear that we can multiply $|x-5| < \delta$ by $|3|$ to get $|3| \cdot |x-5| < |3| \cdot \delta = 3\delta$. Thus we get

Let $x \in \mathbb{R}$ be arbitrary.

Assume $0 < |x-5| < \delta$

$|3| \cdot |x-5| < |3| \cdot \delta = 3\delta$

$|3x - 15| < 3\delta$

.

$|(3x+2) - 17| < \epsilon$

$|f(x) - 17| < \epsilon$ (by hypothesis and the definition of f)

If $0 < |x-5| < \delta$, then $|f(x) - 17| < \epsilon$.

It's clear that we need to define $\delta = \epsilon/3$, so that $3\delta = \epsilon$—which finishes the proof:

Proof:

Assume $f(x) = 3x + 2$. We will show $\lim_{x \to 5} f(x) = 17$.

Let $\epsilon > 0$ be given.

Let $\delta = \epsilon/3$

Let $x \in \mathbb{R}$ be arbitrary.

Assume $0 < |x-5| < \delta$.

$|3| \cdot |x-5| < |3| \cdot \delta = 3\delta$

$|3x - 15| < 3\delta$

$|(3x+2) - 17| < \epsilon$

$|f(x) - 17| < \epsilon$ (by hypothesis and the definition of f)

If $0 < |x-5| < \delta$, then $|f(x) - 17| < \epsilon$.

For all $x \in \mathbb{R}$, if $0 < |x-5| < \delta$, then $|f(x) - 17| < \epsilon$.

There exists $\delta > 0$ such that for all $x \in \mathbb{R}$, if $0 < |x-5| < \delta$, then $|f(x) - 17| < \epsilon$.

For every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x \in \mathbb{R}$, if $0 < |x-5| < \delta$, then $|f(x) - 17| < \epsilon$.

$\lim_{x \to 5} f(x) = 17$.

□

The symbol □ signals the end of a complete proof.

By using the same symbol, x, in two different ways, we have made the proof look a little simpler than it really is. Recall that the statement $f(x) = 3x + 2$ serves to define the function f. It is a customary abbreviation (in a function definition) for the more complete, formal statement "for all $x \in \mathbb{R}$, $f(x) = 3x + 2$".

The symbol "x" in the statement "for all $x \in \mathbb{R}$, $f(x) = 3x + 2$" is called a *local* variable. It has no meaning outside the statement; that is, "for all $x \in \mathbb{R}$, $f(x) = 3x + 2$" is about f, not about x. Any other letter (not already in use) could serve as the needed local variable.
Thus

for all $x \in \mathbb{R}$, $f(x) = 3x + 2$
and for all $t \in \mathbb{R}$, $f(t) = 3t + 2$

have exactly the same meaning.

Part way down in our proof we have the statement "Let $x \in \mathbb{R}$ be arbitrary." This serves to define the *global* variable x for all the steps up to, but not including, the statement "For all $x \in \mathbb{R}$, if $0 < |x - 5| < \delta$, then $|f(x) - 17| < \epsilon$." This global variable x is a particular, fixed element of $\mathbb{R}$. It is not the same x used in the hypothesis to define the function f. The hypothesis stated that $f(x) = 3x + 2$, but this was not (and could not be) a statement about the particular x defined later in the proof. In order to avoid this confusion, we could have written the hypothesis of the proposition as "for all $t \in \mathbb{R}$, $f(t) = 3t + 2$". Using this hypothesis, we can conclude $f(x) = 3x + 2$ about the x defined in the proof: since $f(t) = 3t + 2$ is true for all $t \in \mathbb{R}$, it is true for the x defined later in the proof steps. We have implicitly invoked an inference rule for *using for-all* statements:

Inference Rule Using *for-all* statements: If the statement "for all $t \in A$, $\mathcal{P}(t)$" is true, and if $x \in A$, then $\mathcal{P}(x)$ is true.

In our proof development, we have also used the following inference rules:

Inference Rule Proving *for-all* statements: In order to prove the statement "for all $x \in A$, $\mathcal{P}(x)$", let x be an arbitrarily chosen element of A, then prove that $\mathcal{P}(x)$ is true.

Inference Rule Proving *if-then* statements: In order to prove the statement "if $\mathcal{P}$, then $\mathcal{Q}$", assume that $\mathcal{P}$ is true and show that $\mathcal{Q}$ is true.

Inference Rule Proving *there-exists* statements: In order to prove the statement "there exists $x \in A$ such that $\mathcal{P}(x)$", define the symbol x, and prove $x \in A$ and $\mathcal{P}(x)$ for the x you have defined.

In the proof we have developed, the same phrases are written over and over again. Although this was done to expose the logic of the outlining project, proofs written in texts don't have this redundancy. In order to make our proof look more like a text proof, we can abbreviate it by omitting those lines consisting of formal statements at the top level; that is, *for-all, there-exists, if-then,* and *or* statements. We will do this one line at a time, in order to see why. Consider our complete proof:

Proof:

Assume $f(x) = 3x + 2$. We will show $\lim_{x\to 5} f(x) = 17$.

Let $\epsilon > 0$ be given.

Let $\delta = \epsilon/3$

Let $x \in \mathbb{R}$ be arbitrary.

Assume $0 < |x - 5| < \delta$.

$|3| \cdot |x - 5| < |3| \cdot \delta = 3\delta$

$|3x - 15| < 3\delta$

$|(3x + 2) - 17| < \epsilon$

$|f(x) - 17| < \epsilon$ (by hypothesis and the definition of f)

If $0 < |x - 5| < \delta$, then $|f(x) - 17| < \epsilon$.

For all $x \in \mathbb{R}$, if $0 < |x - 5| < \delta$, then $|f(x) - 17| < \epsilon$.

There exists $\delta > 0$ such that for all $x \in \mathbb{R}$, if $0 < |x - 5| < \delta$, then $|f(x) - 17| < \epsilon$.

For every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x \in \mathbb{R}$, if $0 < |x - 5| < \delta$, then $|f(x) - 17| < \epsilon$.

$\lim_{x\to 5} f(x) = 17$.

□

The formal *if-then* statement just below the lines with white background is proved by precisely those lines. The *if-then* statement itself would not be stated in an informal or narrative proof, since it seems only to say all over again that which has just been said implicitly (in the white background lines preceding it). That is, we know that the *if-then* statement is true, as soon as we have read the lines above it, so it looks as if the *if-then* statement is a mere recapitulation. In an abbreviated proof, it is omitted. When we omit it from our proof, we have the following (the area to which we will refer next is put in the white background—ahead of time) :

Proof:

Assume $f(x) = 3x + 2$. We will show $\lim_{x\to 5} f(x) = 17$.

Let $\epsilon > 0$ be given.

Let $\delta = \epsilon/3$

Let $x \in \mathbb{R}$ be arbitrary.

Assume $0 < |x - 5| < \delta$.

$|3| \cdot |x - 5| < |3| \cdot \delta = 3\delta$

$|3x - 15| < 3\delta$

$|(3x + 2) - 17| < \epsilon$

$|f(x) - 17| < \epsilon$ (by hypothesis and the definition of f)

For all $x \in \mathbb{R}$, if $0 < |x - 5| < \delta$, then $|f(x) - 17| < \epsilon$.

There exists $\delta > 0$ such that for all $x \in \mathbb{R}$, if $0 < |x - 5| < \delta$, then $|f(x) - 17| < \epsilon$.

For every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x \in \mathbb{R}$, if $0 < |x - 5| < \delta$, then $|f(x) - 17| < \epsilon$.

$\lim_{x\to 5} f(x) = 17$.

□

Again, the formal *for-all* statement "for all $x \in \mathbb{R}$, if $0 < |x-1| < \delta$, then $|f(x)-2| < \epsilon$" just after the white background lines seems only to recapitulate what we know from these lines. It is customarily omitted in an abbreviated proof. If we omit it, we get:

Proof:

Assume $f(x) = 3x + 2$. We will show $\lim_{x \to 5} f(x) = 17$.

Let $\epsilon > 0$ be given.

Let $\delta = \epsilon/3$

Let $x \in \mathbb{R}$ be arbitrary.

Assume $0 < |x-5| < \delta$.

$|3| \cdot |x-5| < |3| \cdot \delta = 3\delta$

$|3x - 15| < 3\delta$

$|(3x+2) - 17| < \epsilon$

$|f(x) - 17| < \epsilon$ (by hypothesis and the definition of f)

There exists $\delta > 0$ such that for all $x \in \mathbb{R}$, if $0 < |x-5| < \delta$, then $|f(x)-17| < \epsilon$.

For every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x \in \mathbb{R}$, if $0 < |x-5| < \delta$, then $|f(x) - 17| < \epsilon$.

$\lim_{x \to 5} f(x) = 17$.

□

Similarly, the formal *there-exists* statement "there exists $\delta > 0$ such that for all $x \in \mathbb{R}$, if $0 < |x-1| < \delta$, then $|f(x)-2| < \epsilon$" just after the white background lines recapitulates what we know from these lines, and we delete it to get:

Proof:

Assume $f(x) = 3x + 2$. We will show $\lim_{x \to 5} f(x) = 17$.

Let $\epsilon > 0$ be given.

Let $\delta = \epsilon/3$

Let $x \in \mathbb{R}$ be arbitrary.

Assume $0 < |x-5| < \delta$.

$|3| \cdot |x-5| < |3| \cdot \delta = 3\delta$

$|3x - 15| < 3\delta$

$|(3x+2) - 17| < \epsilon$

$|f(x) - 17| < \epsilon$ (by hypothesis and the definition of f)

For every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x \in \mathbb{R}$, if $0 < |x-5| < \delta$, then $|f(x) - 17| < \epsilon$.

$\lim_{x \to 5} f(x) = 17$.

□

Finally, the formal *for-all* statement "for every $\epsilon > 0$, there exists $\ldots$" is deleted. There are two reasons that justify this deletion. First, as we have seen before, it seems only to repeat what we know to be true by the white background lines preceding it. Second, by the definition of limit, it is equivalent to the statement $\lim_{x \to 5} f(x) = 17$ just below it. In our minds, we identify the statement $\lim_{x \to 5} f(x) = 17$ with its defining condition. We prove the defining condition, but we think of this as proving the limit statement. Thus the definition of limit is not written out explicitly as a proof step, but used implicitly. We justify the last step by mentioning this definition. This gives:

Proof:

Assume $f(x) = 3x + 2$. We will show $\lim_{x \to 5} f(x) = 17$.

Let $\epsilon > 0$ be given.

Let $\delta = \epsilon/3$

Let $x \in \mathbb{R}$ be arbitrary.

Assume $0 < |x - 5| < \delta$.

$|3| \cdot |x - 5| < |3| \cdot \delta = 3\delta$

$|3x - 15| < 3\delta$

$|(3x + 2) - 17| < \epsilon$

$|f(x) - 17| < \epsilon$ (by hypothesis and the definition of f)

Therefore $\lim_{x \to 5} f(x) = 17$ by the definition of limit.

□

Thus all statements that were introduced by the proof outlining procedure and that were of one of the forms *or, for-all, there-exists,* or *if-then* have been deleted to abbreviate the proof. These statements dictated the form of the proof—and once they have done this, need not be retained.

We now change the series of lines to paragraph style, add a few words to soften the abruptness, and explain anything we think the reader might not see:

Proof:

Assume for all $x \in \mathbb{R}$, $f(x) = 3x + 2$. We will show $\lim_{x \to 5} f(x) = 17$. Let $\epsilon > 0$ be given. Let $\delta = \epsilon/3$ and let $x \in \mathbb{R}$ be arbitrary. Then if we assume $0 < |x - 5| < \delta$, we get $|3| \cdot |x - 5| < |3| \cdot \delta = 3\delta$, so that $|3x - 15| < 3\delta$; that is, $|(3x + 2) - 17| < \epsilon$. Therefore $|f(x) - 17| < \epsilon$ by definition of f. Therefore $\lim_{x \to 5} f(x) = 17$ by the definition of limit.

□

In this final version of the proof, we neglect to mention that we are using the hypothesis when we say "by definition of f". The hypothesis being the place where f is defined, we necessarily must be using it—so we needn't say so.

Although the final version of the proof is written in the way that proofs are most often written, it is no more *correct* than the original, long-winded version. It merely involves many things implicitly that in the original version were explicit.[10]

[10] Even in the original, long-winded version, the inference rules were used implicitly. They were not mentioned in the proof itself, but they dictated the form of the proof. In the author's texts mentioned in the introduction, *everything* is done explicitly at first. Furthermore, the statement defining "limit" in the present development is a statement of numerical inequality—contained in an *if-then* statement—contained in a *for-all* statement—contained in a *there-exists* statement—contained in a *for-all* statement. We therefore *begin* our development with a definition that involves 5 levels of nested logical statements. In the more basic texts mentioned in the introduction, on the other hand, statements build slowly in complexity. The early material in these texts—while still developing the proof outlining method—is thus accessible to students with less maturity than that necessary for following our development here. In turn, our development here demands much less maturity than the writing of proofs without the benefit of a method—which is beyond most students' grasp.

A proof is a sequence of statements leading to the conclusion. Each of these statements must either define new symbols or involve only symbols that have been previously defined—either in the hypotheses, or earlier in the proof statements. Consider the variables in the sentences in the proof above. The first sentence "Assume for all $x \in \mathbb{R}$, $f(x) = 3x + 2$." merely copies the hypothesis. The symbol x is a local variable whose *scope* is restricted to the statement itself. It is free to be used again in other ways in the proof, and is used again as a local variable in the next statement "we will show $\lim_{x\to 5} f(x) = 17$". The statement "let $x \in \mathbb{R}$ be given", however, defines the symbol x as a global variable whose scope is the set of statements ending with "$|f(x) - 17| < \epsilon$". This is the set of statements needed to prove the top-level *for-all* statement "for all $x \in \mathbb{R}$, if $0 < |x - 5| < \delta$, then $|f(x) - 17| < \epsilon$." This use of x as a global variable ceases with the statement "$|f(x) - 17| < \epsilon$" just before the *for-all* statement. It is therefore free to be used as a local variable in the *for-all* statement (if the *for-all* statement is included in the proof). It is also used as a local variable in the conclusion "therefore $\lim_{x\to 5} f(x) = 17$" The symbol x is thus used over and over again in the proof in different ways. Although this is customary, it is absolutely essential that one symbol not represent two *different* things *at the same time*. In particular, we can't use a currently defined global variable also as a local variable: we cant talk about all t's, if t has already been defined and is still in use as some particular number. Note that in the final, abbreviated proof, the symbols ϵ and δ each have a single use as a global variable, following their definitions in the proof itself.

In order to illustrate the general breakdown of a proposition into hypotheses and conclusion, we worded Example 1 to have a function definition in the hypotheses. This need not be done explicitly. In the following example, we are asked merely to show that $\lim_{x\to 5}(3x + 2) = 17$.

Example 2 Show that $\lim_{x\to 5}(3x + 2) = 17$.

It is implicit that $3x + 2$ is a function (and not a number, say), since the only things that have limits as x goes to 5 are functions of x. We can change our proof to one based on this phrasing of the proposition (no hypothesis—all conclusion) merely by leaving out the hypothesis and writing $3x + 2$ in place of $f(x)$. We also don't bother to say "we will show $\lim_{x\to 5}(3x + 2) = 17$", since that is obvious. With these changes, we have

Proof:

Let $\epsilon > 0$ be given. Let $\delta = \epsilon/3$ and let $x \in \mathbb{R}$ be given. Then if we assume $0 < |x - 5| < \delta$, we get $|3| \cdot |x - 5| < |3| \cdot \delta = 3\delta$, so that $|3x - 15| < 3\delta$; that is, $|(3x + 2) - 17| < \epsilon$. Therefore $\lim_{x\to 5}(3x + 2) = 17$ by the definition of limit.

□

EXERCISE

1. Let $f(x) = 5x - 3$. Prove $\lim_{x\to 2} f(x) = 7$. Don't do this exercise by copying steps in the proofs given above. Instead, copy the procedure used to get the steps.

Using Definitions and Theorems

In developing a proof outline for the example in the previous section, a statement of the form *number* $<$ *number* presented itself at the bottom of the gap. Since the relation $<$ had not been defined, and since the statement was not one of the basic logic forms (there exists, for all, and so on), the outlining process came to an end at that point. In this section, we will define the relation $<$, in order to make a few important points about proof. Our aim is not to seek to justify algebraic manipulations or computations logically.[11] Generally, our employment of logical reasoning will be restricted to the development of a proof outline. Familiar (and, perhaps, not-so-familiar) algebraic computations will be accepted as part of a proof on the same basis that they have been accepted in mathematics prerequisite for the calculus.

Today, the generally accepted foundation for mathematics is given in terms of sets. Thus everything, including number, is ultimately defined in terms of sets. Our definition of the relation $<$ will be in terms of the set $\mathbb{R}^+$ of positive real numbers. We will assume as an axiom about this set that it is closed under addition and multiplication. That is:

Axiom For all elements a and b in $\mathbb{R}^+$, the sum $a+b$ and product ab are also in $\mathbb{R}^+$.

Definition For numbers a and b in the set $\mathbb{R}$, define $a<b$ iff there exists $d \in \mathbb{R}^+$ such that $a+d=b$.

Proposition 1 (Transitivity of $<$) For real numbers $a, b,$ and c, if $a<b$ and $b<c$, then $a<c$.

Proof:

Assume $a<b$ and $b<c$ for $a,b,c \in \mathbb{R}$. We will show $a<c$.

.

.

.

$a<c$

The next step in the outlining process is to put the definition of $a<c$ as the line immediately preceding this statement:

[11] All mathematics is based on logical principles. The familiar algebraic computations are logically based on properties of the set $\mathbb{R}$ of real numbers—the axioms for its being a complete, totally-ordered field. Some of these axioms, notably the commutative, associative, and distributive properties of addition and multiplication, are normally shown to be the basis of algebraic manipulation in school mathematics.

Proof:

Assume $a < b$ and $b < c$ for $a, b, c \in \mathbb{R}$. We will show $a < c$.
.
.
.
There exists $d \in \mathbb{R}^+$ such that $a + d = c$.
$a < c$

The rule for proving *there-exists* statements dictates that we need to define the element d, (to show where this is done, we rough in the line "let $d = \dots$"), and then show for this d that $d \in \mathbb{R}^+$ and that $a + d = c$. This leaves two gaps in the proof—above the two things we need to show:

Proof:

Assume $a < b$ and $b < c$ for $a, b, c \in \mathbb{R}$. We will show $a < c$.
.
Let $d = \dots$
.
.
$d \in \mathbb{R}^+$
.
.
$a + d = c$
There exists $d \in \mathbb{R}^+$ such that $a + d = c$.
$a < c$

We can work up from the bottom no longer, so we now use the hypotheses. By the first hypothesis, $a < b$, we can add the defining condition "there exists $d \in \mathbb{R}^+$ such that $a + d = b$" to the proof as the second line. This *there-exists* statement will serve to define the element d for the rest of the proof. Also, from this statement we may infer the statements $d \in \mathbb{R}^+$ and $a + d = b$ about this d. Formally, we say that the rule for using *there-exists* statements allows us to do this. Informally, we merely take the *there-exists* statement at its word. Either way, d is defined by the *there-exists* statement. However, this may not be the same d that we want later in the proof, for the lines that follow from "let $d = \dots$" (the statement we have roughed in). In the proof, we therefore must use a symbol other than d in the condition defining the relation $a < b$. We'll use d'. Similarly, we need to use some symbol such as d'' in the defining condition for the second hypothesis:

Proof:

Assume $a < b$ and $b < c$ for $a, b, c \in \mathbb{R}$. We will show $a < c$.
There exists $d' \in \mathbb{R}^+$ such that $a + d' = b$ (by the first hypothesis).
There exists $d'' \in \mathbb{R}^+$ such that $b + d'' = c$ (by the second hypothesis).
Let $d = \dots$
.
.
$d \in \mathbb{R}^+$
.
.
$a + d = c$

There exists $d \in \mathbb{R}^+$ such that $a + d = c$.
$a < c$

This ends the proof outlining process, and leaves us with a well-defined (computational) task. We need to show $a + d = c$ from the equations $a + d' = b$ and $b + d'' = c$, and we're free to define d any way we want, in order to do this. If we add d'' to both sides of $a + d' = b$, we get

$$a + d' + d'' = b + d''.$$

Since $b + d'' = c$, we get

$$a + d' + d'' = c.$$

It seems clear that we need to define $d = d' + d''$. Since $\mathbb{R}^+$ is closed under addition, we have $d \in \mathbb{R}^+$. This completes the proof:

Proof:

Assume $a < b$ and $b < c$ for $a, b, c \in \mathbb{R}$. We will show $a < c$.
There exists $d' \in \mathbb{R}^+$ such that $a + d' = b$ (by the first hypothesis).
There exists $d'' \in \mathbb{R}^+$ such that $b + d'' = c$ (by the second hypothesis).
Let $d = d' + d''$.
Then $d \in \mathbb{R}^+$, since $\mathbb{R}^+$ is closed under addition.
Adding d'' to the equation in line 2 gives: $a + d' + d'' = b + d''$
Substituting line 3 gives: $a + d' + d'' = c$.
Therefore $a + d = c$, by the definition of d.
There exists $d \in \mathbb{R}^+$ such that $a + d = c$.
$a < c$

□

The next-to-last line can be deleted, since it recapitulates the lines before it (starting with "let $d = d' + d''$ "). Although lines 2 and 3 are expressed in formal language, they can't be deleted, since they serve to define the numbers d' and d''. This gives:

Proof:

Assume $a < b$ and $b < c$ for $a, b, c \in \mathbb{R}$. We will show $a < c$.
There exists $d' \in \mathbb{R}^+$ such that $a + d' = b$ (by the first hypothesis).
There exists $d'' \in \mathbb{R}^+$ such that $b + d'' = c$ (by the second hypothesis).
Let $d = d' + d''$.
Then $d \in \mathbb{R}^+$, since $\mathbb{R}^+$ is closed under addition.
Adding d'' to the equation in line 2 gives: $a + d' + d'' = b + d''$
Substituting line 3 gives: $a + d' + d'' = c$.
Therefore $a + d = c$, by the definition of d.
Therefore $a < c$, by the definition of $<$.

□

The notation $a > b$ means exactly the same as $b < a$. The use of one or the other might depend, not on mathematics, but on its linguistic context. For example, we might wish to make "a" the subject and "is greater than b" the predicate, or we might wish to make "b" the subject and "is less than a" the predicate. In formal mathematics, however, everything[12] is defined in terms of previously defined things. A formal definition of the relation $>$, in terms of the previously defined relation $<$ will serve to make a point.

Definition For $a, b \in \mathbb{R}$, define $a > b$ iff $b < a$.

Proposition 2 (Transitivity of $>$) For real numbers q, r, and s, if $q > r$ and $r > s$, then $q > s$.

A proof of Proposition 2 begins, as usual, with the identification of hypotheses and conclusion, and the conclusion as the last line:

Proof:

Assume $q > r$ and $r > s$ for $q, r, s \in \mathbb{R}$. We will show $q > s$.
.
.
.
$q > s$

At the top level, the conclusion is of the form *number* $>$ *number*. The meaning of this is given by the definition of $>$, which gives us the next-to-last line:

Proof:

Assume $q > r$ and $r > s$ for $q, r, s \in \mathbb{R}$. We will show $q > s$.
.
.
$s < q$
$q > s$

Now the statement at the bottom of the gap is of the form *number* $<$ *number*. The normal development of the proof outline method would have us use the definition of $<$ to write the preceding line. However, we now also have a theorem (Proposition 1) about $<$, which we might use. Supposedly not knowing which is best at this stage, we could abandon the bottom-up development and use the information from the hypotheses:

Proof:

Assume $q > r$ and $r > s$ for $q, r, s \in \mathbb{R}$. We will show $q > s$.
$r < q$ (by the first hypothesis and the definition of $>$)
$s < r$ (by the second hypothesis and the definition of $>$)
.
.
.
$s < q$
$q > s$

[12] except the fewest possible undefined terms, with properties given by axioms.

It's clear now that the gap can be closed merely by appealing to Proposition 1. The next-to-last line $s < q$ follows from the second and third lines, by Proposition 1. This gives the complete proof:

Proof:

Assume $q > r$ and $r > s$ for $q, r, s \in \mathbb{R}$. We will show $q > s$.
$r < q$ (by the first hypothesis and the definition of $>$)
$s < r$ (by the second hypothesis and the definition of $>$)
$s < q$ (by Proposition 1)
$q > s$ (by the definition of $>$)

□

Consider the stage of the proof just before we employed Proposition 1:

Proof:

Assume $q > r$ and $r > s$ for $q, r, s \in \mathbb{R}$. We will show $q > s$.
$r < q$ (by the first hypothesis and the definition of $>$)
$s < r$ (by the second hypothesis and the definition of $>$)
.
.
$s < q$
$q > s$

The second and third lines are the hypotheses of Proposition 1. The statement at the bottom of the gap is the conclusion of Proposition 1. If we had had failed to *use* Proposition 1, but continued the proof outlining process using the definition of $<$, we would merely have repeated the proof of Proposition 1, now within the proof of Proposition 2. In principle, it's not necessary to use theorems to prove other theorems. Everything must come from definitions. Practically, however, we couldn't possibly write out complete proofs, if we didn't appeal to theorems.

The use of theorems to prove other theorems is an essential shortcut. But it is, nevertheless, a shortcut. Proof at its most basic level depends on definitions. The easiest proofs—the ones a mathematician would call “straightforward” are the proofs for which the outlining method produces a complete proof.

While the outlining method doesn't reveal creative steps need to fill a gap in a proof, it at least gets one going. Furthermore, as the footnote on page vi attests, it is powerful enough to enable students to do very well in many advanced courses.

The material in this section has illustrated the following formal rule:

Inference Rule Using *there-exists* statements: The statement “there exists $x \in A$ such that $\mathcal{P}(x)$” defines the symbol x as a global variable and allows us to use the statements $x \in A$ and $\mathcal{P}(x)$ about x.

EXERCISE

2. Use the proof outlining method and the definition of $<$ to show that one can add the same number to each side of an inequality. That is, prove that for $a, b, c \in \mathbb{R}$, if $a < b$, then $a + c < b + c$. Note that adding the same number to both sides of an *equality* can be justified by substitution: if $a = b$, then substituting b for a in $a + c = a + c$ produces $a + c = b + c$, but this doesn't work for inequality.

Limit Theorems

With the proof of Theorem 1, we will begin the process of abbreviation from the start of the development of a proof—rather than developing a long-winded version to be abbreviated later. The theorem assumes that the functions f and g have a common domain D.

Definition If $f\colon D \to \mathbb{R}$ and $g\colon D \to \mathbb{R}$, define the function $f + g\colon D \to \mathbb{R}$ by $(f+g)(x) = f(x) + g(x)$.

Theorem 1 If $\lim\limits_{x\to c} f(x) = L$ and $\lim\limits_{x\to c} g(x) = M$, then $\lim\limits_{x\to c} (f+g)(x) = L + M$.

We first state the hypotheses and conclusion at the top of the proof, and then write the conclusion as the last line.

Proof:

Assume $\lim\limits_{x\to c} f(x) = L$ and $\lim\limits_{x\to c} g(x) = M$. We will show $\lim\limits_{x\to c} (f+g)(x) = L + M$.

.
.
.

$\lim\limits_{x\to c} (f+g)(x) = L + M$.

The condition equivalent to $\lim\limits_{x\to c} (f+g)(x) = L + M$, by the definition of limit, is now put as the statement immediately preceding the conclusion:

Proof:

Assume $\lim\limits_{x\to c} f(x) = L$ and $\lim\limits_{x\to c} g(x) = M$. We will show $\lim\limits_{x\to c} (f+g)(x) = L + M$.

.
.
.

For every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x \in D$, if $0 < |x - c| < \delta$, then $|(f+g)(x) - [L+M]| < \epsilon$.

$\lim\limits_{x\to c} (f+g)(x) = L + M$.

The rule for proving *for-all* statements dictates that the following statements (with white background) be added to the proof:

Proof:

Assume $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$. We will show $\lim_{x \to c} (f+g)(x) = L + M$.

Let $\epsilon > 0$ be given.

.

.

.

There exists $\delta > 0$ such that for all $x \in D$, if $0 < |x - c| < \delta$, then $|(f+g)(x) - [L+M]| < \epsilon$.

For every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x \in D$, if $0 < |x - c| < \delta$, then $|(f+g)(x) - [L+M]| < \epsilon$.

$\lim_{x \to c} (f+g)(x) = L + M$.

After adding the statements with white background, we can delete the formal *for-all* statement "for every $\epsilon > 0 \dots$" just below them. Note that this has the effect of replacing the *for-all* statement with the two statements—one at the top and one at the bottom of the gap. If one were developing the proof on scrap paper, one could merely erase "for every $\epsilon > 0$," from the *for-all* statement and put "let $\epsilon > 0$ be given" at the top of the gap—which would leave the *there-exists* statement at the bottom of the gap. No need to capitalize the "t". It will be erased in the next step.

Proof:

Assume $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$. We will show $\lim_{x \to c} (f+g)(x) = L + M$.

Let $\epsilon > 0$ be given.

.

.

.

There exists $\delta > 0$ such that for all $x \in D$, if $0 < |x - c| < \delta$, then $|(f+g)(x) - [L+M]| < \epsilon$.

$\lim_{x \to c} (f+g)(x) = L + M$.

The statement at the bottom of the gap is a *there-exists* statement. The rule for proving such statements says that we must define δ in the proof steps, and prove the lower-level statement "for all $x \in D$, $\dots$" that involves this δ. We therefore add the statement "define $\delta = \dots$" in the gap, and erase the words "there exists $\delta > 0$ such that" to get the new statement at the bottom of the gap:

Proof:

Assume $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$. We will show $\lim_{x \to c} (f+g)(x) = L + M$.

Let $\epsilon > 0$ be given.

.

Define $\delta = \dots$

.

.

.

For all $x \in D$, if $0 < |x - c| < \delta$, then $|(f+g)(x) - [L+M]| < \epsilon$.

$\lim_{x \to c} (f+g)(x) = L + M$.

The *for-all* statement at the bottom of the gap dictates that the line "let $x \in D$ be arbitrary" be added at the top of the gap, and that "for all $x \in D$" be erased to leave an *if-then* statement at the bottom:

Proof:

Assume $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$. We will show $\lim_{x \to c}(f+g)(x) = L + M$.
Let $\epsilon > 0$ be given.
.
Define $\delta = \dots$
Let $x \in D$ be arbitrary.
.
.
.
If $0 < |x - c| < \delta$, then $|(f+g)(x) - [L+M]| < \epsilon$.
$\lim_{x \to c}(f+g)(x) = L + M$.

The *if-then* statement dictates that we add "assume $0 < |x - c| < \delta$" at the top of the gap an erase the words "if $0 < |x - c| < \delta$, then" from the statement at the bottom. This gives:

Proof:

Assume $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$. We will show $\lim_{x \to c}(f+g)(x) = L + M$.
Let $\epsilon > 0$ be given.
.
Define $\delta = \dots$
Let $x \in D$ be arbitrary.
Assume $0 < |x - c| < \delta$.
.
.
.
$|(f+g)(x) - [L+M]| < \epsilon$.
$\lim_{x \to c}(f+g)(x) = L + M$.

It is common to contract the two lines just above the gap into a single line: "assume $0 < |x - c| < \delta$ for an arbitrary $x \in D$." Thus instead of applying the rules for proving *for-all* statements and *if-then* statements one at a time as we have done above, we do it all at once with a contracted rule for proving statements of the very common form *for-all-if-then*. This formal rule is analogous to the way we prove a theorem after deciding on its informal hypotheses/conclusion interpretation. The rule, illustrated by statements in our proof, allows us to infer the *for-all-if-then* statement below from the two statements above it:

Assume $0 < |x - c| < \delta$ for an arbitrary $x \in D$.
.
.
.
$|(f + g)(x) - [L + M]| < \epsilon$
For all $x \in D$, if $0 < |x - c| < \delta$, then $|(f + g)(x) - [L + M]| < \epsilon$.

In general, we have the following rule:

Inference Rule Proving *for-all-if-then* statements: In order to prove the statement "for all $t \in A$, if $\mathcal{P}(t)$, then $\mathcal{Q}(t)$", assume $\mathcal{P}(t)$ for an arbitrary $t \in A$, then prove that $\mathcal{Q}(t)$ is true.

Incorporating this in our outline, we have:

Proof:

Assume $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$. We will show $\lim_{x \to c} (f + g)(x) = L + M$.
Let $\epsilon > 0$ be given.
.
Define $\delta = \dots$
Assume $0 < |x - c| < \delta$ for an arbitrary $x \in D$.
.
.
.
$|(f + g)(x) - [L + M]| < \epsilon$.
$\lim_{x \to c} (f + g)(x) = L + M$.

We can continue to work up from the bottom of the gap, if we use the definition of the function $f + g$:

Proof:

Assume $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$. We will show $\lim_{x \to c} (f + g)(x) = L + M$.
Let $\epsilon > 0$ be given.
.
Define $\delta = \dots$
Assume $0 < |x - c| < \delta$ for an arbitrary $x \in D$.
.
.
$|(f(x) + g(x) - [L + M]| < \epsilon$.
$|(f + g)(x) - [L + M]| < \epsilon$.
$\lim_{x \to c} (f + g)(x) = L + M$.

It is time to use the hypotheses. From the first hypothesis, $\lim_{x \to c} f(x) = L$, we infer the defining condition: "for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x \in D$, if $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$". This statement could be written at the top of the first gap—just below the statement "Let $\epsilon > 0$ be given." This latter statement, however, defines ϵ as a global variable for use in the set of statements up to the next-to-last line of the proof, so we can't use the same symbol ϵ as a local variable within this set of statements. Since ϵ has been defined as a specific number by the statement "let $\epsilon > 0$ be given.", we can't talk about "all ϵ" or "every ϵ" in the very next statement. We therefore use some other symbol such as ϵ' in the defining condition. Also, since the δ given by the defining condition may not be the same δ we need to define later (it won't be) we use the symbol δ' in the defining condition. We thus get the following:

Proof:

Assume $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$. We will show $\lim_{x\to c}(f+g)(x) = L+M$.

Let $\epsilon > 0$ be given.

By the first hypothesis, we have for all $\epsilon' > 0$, there exists $\delta' > 0$ such that for all $x \in D$, if $0 < |x-c| < \delta'$, then $|f(x) - L| < \epsilon'$.

By the second hypothesis, we have for all $\epsilon'' > 0$, there exists $\delta'' > 0$ such that for all $x \in D$, if $0 < |x-c| < \delta''$, then $|g(x) - M| < \epsilon''$.

.

Define $\delta = \ldots$

Assume $0 < |x-c| < \delta$ for an arbitrary $x \in D$.

.

.

$|(f(x) + g(x) - [L+M]| < \epsilon$.

$|(f+g)(x) - [L+M]| < \epsilon$.

$\lim_{x\to c}(f+g)(x) = L+M$.

At this point of the proof outline, it remains to define δ, and for that δ to show that $|[f(x) + g(x)] - [L+M]| < \epsilon$ follows from $0 < |x-c| < \delta$. We can use the validity of two *for-all* statements: "for all $\epsilon' > 0, \ldots$" and "for all $\epsilon'' > 0, \ldots$". In order to do this, we need to apply the statement to *specific* values of the variables ϵ' and ϵ'' (by the rule for using *for-all* statements.) That is, ϵ' and ϵ'' must be defined in terms of the only other (non-local) variable we have in the proof at that stage, namely ϵ.

In order to see how to do this, we work up from the bottom—now using computations: $[f(x) + g(x)] - [L+M] = [f(x) - L] + [g(x) - M]$. By the proof lines obtained from the hypotheses, we can make $[f(x) - L]$ and $[g(x) - M]$ as small as we want: less than ϵ' and ϵ'', respectively, where we can choose ϵ' and ϵ'' any way we want. If we make $f(x) - L$ less than $\epsilon/2$ and also $g(x) - M$ less than $\epsilon/2$, then certainly their sum will be less than ϵ. Actually, we need to make the *absolute value* of their sum less than ϵ. The way to handle absolute values is with the *triangle inequality*, which states that for all numbers a and b, $|a+b| \leq |a| + |b|$.

Continuing to work up from the bottom, and applying the triangle inequality, we get:

Assume $0 < |x-c| < \delta$ for an arbitrary $x \in D$.

.

.

.

$|[f(x) + g(x)] - [L+M]| =$
$|[f(x) - L] + [g(x) - M]| \leq |f(x) - L| + |g(x) - M| < \epsilon/2 + \epsilon/2 = \epsilon$.

Therefore $|(f+g)(x) - [L+M]| < \epsilon$.

$\lim_{x\to c}(f+g)(x) = L+M$.

We now apply (using the rule for using *for-all* statements) the truth of the statement "for all $\epsilon' > 0, \ldots$" to the specific value $\epsilon' = \epsilon/2$, to get the added line "there exists $\delta' > 0 \ldots$"—just below the white area in the proof:

Proof:

Assume $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$. We will show $\lim_{x\to c}(f+g)(x) = L+M$.
Let $\epsilon > 0$ be given.

By the first hypothesis, we have for all $\epsilon' > 0$, there exists $\delta' > 0$ such that for all $x \in D$, if $0 < |x-c| < \delta'$, then $|f(x) - L| < \epsilon'$.

There exists $\delta' > 0$ such that for all $x \in D$, if $0 < |x-c| < \delta'$, then $|f(x) - L| < \epsilon/2$.

By the second hypothesis, we have for all $\epsilon'' > 0$, there exists $\delta'' > 0$ such that for all $x \in D$, if $0 < |x-c| < \delta''$, then $|g(x) - M| < \epsilon''$.

.

Define $\delta = \ldots$

Assume $0 < |x-c| < \delta$ for an arbitrary $x \in D$.

.

The line in the white area above is merely the defining condition for the hypothesis $\lim_{x\to c} f(x) = L$. We have applied the truth of this statement about *all* ϵ' to get the *there-exists* statement just below it —but we think of this as applying the *hypothesis* $\lim_{x\to c} f(x) = L$ to infer the *there-exists* statement. That is, in our minds we identify the defining condition with the thing defined. For this reason, the explicit repetition of defining conditions are not included in proofs. We therefore delete it to abbreviate the proof:

Proof:

Assume $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$. We will show $\lim_{x\to c}(f+g)(x) = L+M$.
Let $\epsilon > 0$ be given.

By the first hypothesis, there exists $\delta' > 0$ such that for all $x \in D$, if $0 < |x-c| < \delta'$, then $|f(x) - L| < \epsilon/2$.

By the second hypothesis, we have for all $\epsilon'' > 0$, there exists $\delta'' > 0$ such that for all $x \in D$, if $0 < |x-c| < \delta''$, then $|g(x) - M| < \epsilon''$.

.

Define $\delta = \ldots$

Similarly, instead of writing down the defining condition for the second hypothesis, we apply it to the specific value $\epsilon'' = \epsilon/2$ and write only the *there-exists* statement that we infer:

Proof:

Assume $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$. We will show $\lim_{x\to c}(f+g)(x) = L+M$.

Let $\epsilon > 0$ be given.

By the first hypothesis, there exists $\delta' > 0$ such that for all $x \in D$, if $0 < |x-c| < \delta'$, then $|f(x) - L| < \epsilon/2$.

By the second hypothesis, there exists $\delta'' > 0$ such that for all $x \in D$, if $0 < |x-c| < \delta''$, then $|g(x) - M| < \epsilon/2$.

.

Define $\delta = \ldots$

Assume $0 < |x-c| < \delta$ for an arbitrary $x \in D$.

.

.

The conclusions, $|f(x) - L| < \epsilon/2$ and $|g(x) - M| < \epsilon/2$, of the *for-all-if-then* statements within these top level *there-exists* statements are just what we need to close the gap in the proof. The following rules tells us what is necessary to infer the conclusions.

Inference Rule Using *if-then* statements: If the statement "if $\mathcal{P}$, then $\mathcal{Q}$" is true, and if $\mathcal{P}$ is true, then we may infer that $\mathcal{Q}$ is true.

Inference Rule Using *for-all-if-then* statements: If the statement "for all $t \in A$, if $\mathcal{P}(t)$, then $\mathcal{Q}(t)$" is true, and if $\mathcal{P}(x)$ is true for some $x \in A$, then we may infer that $\mathcal{Q}(x)$ is true.

By the rule for using *for-all-if-then* statements, it suffices to show $0 < |x-c| < \delta'$ and $0 < |x-c| < \delta''$ for the x defined by the step where we assume $0 < |x-c| < \delta$ for an arbitrary $x \in D$. Thus by using the hypotheses, we can back up one more step from the bottom of the gap, to get:

Proof:

Assume $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$. We will show $\lim_{x\to c}(f+g)(x) = L+M$.

Let $\epsilon > 0$ be given.

By the first hypothesis, there exists $\delta' > 0$ such that for all $x \in D$, if $0 < |x-c| < \delta'$, then $|f(x) - L| < \epsilon/2$.

By the second hypothesis, there exists $\delta'' > 0$ such that for all $x \in D$, if $0 < |x-c| < \delta''$, then $|g(x) - M| < \epsilon/2$.

Define $\delta = \ldots$

Assume $0 < |x-c| < \delta$ for an arbitrary $x \in D$.

.

.

$0 < |x-c| < \delta'$

$0 < |x-c| < \delta''$

Therefore $|[f(x) + g(x)] - [L+M]| =$
$|[f(x) - L] + [g(x) - M]| \le |f(x) - L| + |g(x) - M| < \epsilon/2 + \epsilon/2 = \epsilon$.

Therefore $|f(x) + g(x) - [L+M]| < \epsilon$.

$|(f+g)(x) - [L+M]| < \epsilon$
$\lim_{x\to c}(f+g)(x) = L+M.$

We need to define δ in such a way as to make $0 < |x-c| < \delta'$ and $0 < |x-c| < \delta''$ both true. The way to do this is to make δ the smaller of the two numbers δ' and δ''. Then if $|x-c|$ is less than δ, it must be less than both δ' and δ''. This gives:

Proof:

Assume $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$. We will show $\lim_{x\to c}(f+g)(x) = L+M$.

Let $\epsilon > 0$ be given.

By the first hypothesis, there exists $\delta' > 0$ such that for all $x \in D$, if $0 < |x-c| < \delta'$, then $|f(x) - L| < \epsilon/2$.

By the second hypothesis, there exists $\delta'' > 0$ such that for all $x \in D$, if $0 < |x-c| < \delta''$, then $|g(x) - M| < \epsilon/2$.

Define δ to be the minimum of δ' and δ''.

Assume $0 < |x-c| < \delta$ for an arbitrary $x \in D$.

Then $0 < |x-c| < \delta'$ by the definition of δ'.

Also $0 < |x-c| < \delta''$ by the definition of δ''

Therefore $|f(x) + g(x) - [L+M]| =$
$|[f(x) - L] + [g(x) - M]| \le |f(x) - L| + |g(x) - M| < \epsilon/2 + \epsilon/2 = \epsilon.$

Therefore $|f(x) + g(x) - [L+M]| < \epsilon.$

$|(f+g)(x) - [L+M]| < \epsilon$

$\lim_{x\to c}(f+g)(x) = L+M.$

□

The next-to-last line contracts the previous line. It is not usually thought necessary to do this. We can delete the next-to-last line:

Proof:

Assume $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$. We will show $\lim_{x\to c}(f+g)(x) = L+M$.

Let $\epsilon > 0$ be given.

By the first hypothesis, there exists $\delta' > 0$ such that for all $x \in D$, if $0 < |x-c| < \delta'$, then $|f(x) - L| < \epsilon/2$.

By the second hypothesis, there exists $\delta'' > 0$ such that for all $x \in D$, if $0 < |x-c| < \delta''$, then $|g(x) - M| < \epsilon/2$.

Define δ to be the minimum of δ' and δ''.

Assume $0 < |x-c| < \delta$ for an arbitrary $x \in D$.

Then $0 < |x-c| < \delta'$ by the definition of δ'.

Also $0 < |x-c| < \delta''$ by the definition of δ''

Therefore $|f(x) + g(x) - [L+M]| =$
$|[f(x) - L] + [g(x) - M]| \le |f(x) - L| + |g(x) - M| < \epsilon/2 + \epsilon/2 = \epsilon.$

Therefore $\lim_{x\to c}(f+g)(x) = L+M$, by the definition of limit.

□

Definition For a function $f\colon D \to \mathbb{R}$ and $k \in \mathbb{R}$, define the function $(kf)\colon D \to \mathbb{R}$ by $(kf)(x) = k(f(x))$. We use the expression $kf(x)$ to mean either of the two equivalent expressions $(kf)(x)$ or $k(f(x))$.

Theorem 2 If $\lim_{x\to c} f(x) = L$ and k is any real number, then $\lim_{x\to c} kf(x) = kL$.

We again state the hypotheses and conclusion at the top of the proof, and then write the conclusion as the last line.

Proof:

Assume $\lim_{x\to c} f(x) = L$ and $k \in \mathbb{R}$. We will show $\lim_{x\to c} kf(x) = kL$

.

.

.

$\lim_{x\to c} kf(x) = kL$

The defining condition for the conclusion is the statement "For every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x \in D$, if $0 < |x - c| < \delta$, then $|kf(x) - kL| < \epsilon$." Without writing this in the proof (you may want to write it on scrap paper), we insert, at the top and bottom of the gap, the steps needed to prove it:

Proof:

Assume $\lim_{x\to c} f(x) = L$ and $k \in \mathbb{R}$. We will show $\lim_{x\to c} kf(x) = kL$

Let $\epsilon > 0$ be given.

.

.

.

There exists $\delta > 0$ such that for all $x \in D$, if $0 < |x - c| < \delta$, then $|kf(x) - kL| < \epsilon$.

$\lim_{x\to c} kf(x) = kL$

The added lines outline the proof of the statement "For every $\epsilon > 0$, $\ldots$" that gives the defining condition for $\lim_{x\to c} kf(x) = kL$. We consider, therefore, that the added lines outline the proof of the statement $\lim_{x\to c} kf(x) = kL$—since it is equivalent to its defining condition. We now replace the *there-exists* statement at the bottom of the gap with the statements needed to prove it; that is, we rough in the definition of δ, and erase all but the lower-level *for-all-if-then* statement at the bottom of the gap.

Proof:

Assume $\lim_{x\to c} f(x) = L$ and $k \in \mathbb{R}$. We will show $\lim_{x\to c} kf(x) = kL$
Let $\epsilon > 0$ be given.
.
Let $\delta = \ \ldots$
.
.
.
for all $x \in D$, if $0 < |x - c| < \delta$, then $|kf(x) - kL| < \epsilon$.
$\lim_{x\to c} kf(x) = kL$

In a similar way, we replace the *for-all-if-then* statement at the bottom of the gap with the steps necessary to prove it.

Proof:

Assume $\lim_{x\to c} f(x) = L$ and $k \in \mathbb{R}$. We will show $\lim_{x\to c} kf(x) = kL$
Let $\epsilon > 0$ be given.
.
Let $\delta = \ \ldots$
Assume $0 < |x - c| < \delta$ for an arbitrary $x \in D$.
.
.
.
$|kf(x) - kL| < \epsilon$
$\lim_{x\to c} kf(x) = kL$

There are now no more steps that are dictated by working up from the bottom of the gap, and it is time to use the hypotheses. Instead of writing the defining condition in the steps of the proof, we put it too on scrap paper: "for all $\epsilon' > 0$, there exists $\delta' > 0$ such that for all $x \in D$, if $0 < |x - c| < \delta'$, then $|f(x) - L| < \epsilon'$". We want $|kf(x) - kL| < \epsilon$; that is, $|k||f(x) - L| < \epsilon$; that is, $|f(x) - L| < \epsilon/|k|$. The defining condition is true for all values of $\epsilon' > 0$. We apply this to the value $\epsilon' = \epsilon/|k|$:

Proof:

Assume $\lim_{x\to c} f(x) = L$ and $k \in \mathbb{R}$. We will show $\lim_{x\to c} kf(x) = kL$
Let $\epsilon > 0$ be given.
There exists $\delta' > 0$ such that for all $x \in D$, if $0 < |x - c| < \delta'$, then $|f(x) - L| < \epsilon/|k|$.
Let $\delta = \ \ldots$
Assume $0 < |x - c| < \delta$ for an arbitrary $x \in D$.
.
.
.
$|kf(x) - kL| < \epsilon$
$\lim_{x\to c} kf(x) = kL$

The value of δ that we need is just δ'. We could say "let $\delta = \delta'$, and keep the steps as they are outlined in the proof above. However, it is better not to use two symbols where one will do—so we use only δ:

Proof:

Assume $\lim_{x\to c} f(x) = L$ and $k \in \mathbb{R}$. We will show $\lim_{x\to c} kf(x) = kL$

Let $\epsilon > 0$ be given.

There exists $\delta > 0$ such that for all $x \in D$, if $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon/|k|$.

Assume $0 < |x - c| < \delta$ for an arbitrary $x \in D$.

.

.

.

$|kf(x) - kL| < \epsilon$

$\lim_{x\to c} kf(x) = kL$

It is necessary to define δ in the proof steps, but the *there-exists* statement does this for us (when $\delta = \delta'$).

We observe that the rule for using a *for-all-if-then* statement bridges the gap, and we therefore have a complete proof. In proofs that explicitly contain all logical steps and all justifications for these steps, the steps themselves can be listed as stark assertions. It will be possible to understand such a proof, because of its completeness. In abbreviated proofs, it is best to add a few words that say where things come from. We do this for the proof above:

Proof:

Assume $\lim_{x\to c} f(x) = L$ and $k \in \mathbb{R}$. We will show $\lim_{x\to c} kf(x) = kL$. Let $\epsilon > 0$ be given. Then by hypothesis, there exists $\delta > 0$ such that for all $x \in D$, if $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon/|k|$. If we assume $0 < |x - c| < \delta$ for an arbitrary $x \in D$, then $|f(x) - L| < \epsilon/|k|$, so that $|k||f(x) - L| < \epsilon$; that is, $|kf(x) - kL| < \epsilon$. This shows that $\lim_{x\to c} kf(x) = kL$, by the definition of limit.

□

Free Variables

In some texts, the definition of limit is given as something like the following:

Definition For a function $f\colon D \to \mathbb{R}$ and a number $c \in D$, define $\lim_{x\to c} f(x) = L$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that if $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$.

In this form of the definition, the symbol x appears as a local variable in "$\lim_{x\to c} f(x) = L$", but then again as a global variable without previous definition. The rule for this use is that any such symbol represents an element arbitrarily chosen from a domain that makes sense. In the definition above, the domain is the set D—the domain of the function f. Thus the implication "if $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$" is really an abbreviation of the *for-all-if-then* statement "for all $x \in D$, if $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$". When x is used without prior definition, and when it is not *quantified* (with a phrase "for all x" or "there exists x"), then x is called a *free* *variable*. The rules for working with free variables are the same as the rules for working with the implicit *for-all* statements. In these notes such variables are explicitly quantified.

Definition For a function $f\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and a pair $(a,b)\in\mathbb{R}\times\mathbb{R}$, define $\lim_{(x,y)\to(a,b)} f(x,y)=L$ if for all $\epsilon>0$, there exists $\delta>0$ such that for all $x,y\in\mathbb{R}$, if $0<(x-a)^2+(y-b)^2<\delta$, then $|f(x,y)-L|<\epsilon$.

EXERCISES

3. If $f(x,y)=2x+3y$, prove that $\lim_{(x,y)\to(1,2)} f(x,y)=8$.

4. Outline the proof of the following theorem (the product rule). Then find the proof of the same theorem in some text, to help with the technical steps needed to fill the gap

Theorem 3 If $\lim_{x\to c} f(x)=L$ and $\lim_{x\to c} g(x)=M$, then $\lim_{x\to c}[f(x)\cdot g(x)]=L\cdot M$.

Continuity

On page 1, we stated that all mathematics could be expressed with statements of just 7 types: *for all*, *there exists*, *if-then*, *or*, *and*, *not,* and *iff*, and that the method of proof outlining reflected inference rules for proving and using statements of the first 4 types. We want to mention here statements of the last 3 types.

A formal rule for proving *and* statements says that if $\mathcal{P}$ is true and if $\mathcal{Q}$ is also true, then the statement "$\mathcal{P}$ and $\mathcal{Q}$" is true. In a formal proof, the justification for a step "$\mathcal{P}$ and $\mathcal{Q}$" would cite the rule for proving *and* statements and the numbers of the previous steps with the statements $\mathcal{P}$ and $\mathcal{Q}$. In an informal proof, there is no distinction between the single mathematical *and* statement "$\mathcal{P}$ and $\mathcal{Q}$" and the English language assertion that $\mathcal{P}$ and $\mathcal{Q}$ are both (separately) true. Therefore, the rules for proving and using *and* statements do not help us much with the *form* of the argument of an informal proof—the function of the first 4 rules of inference in proof outlining.

The statement "$\mathcal{P}$ iff $\mathcal{Q}$" means that $\mathcal{P}$ is *equivalent* to $\mathcal{Q}$. We have already applied the rule for using equivalence (which states that a mathematical statement may be substituted for an equivalent one) in substituting a defining condition for the thing defined. "$\mathcal{P}$ iff $\mathcal{Q}$" means the same as "(if $\mathcal{P}$, then $\mathcal{Q}$) and (if $\mathcal{Q}$, then $\mathcal{P}$)". The method of proving that $\mathcal{P}$ is equivalent to $\mathcal{Q}$ depends on the rules for proving *and* and *if-then* statements: assume $\mathcal{P}$ and show $\mathcal{Q}$, then assume $\mathcal{Q}$ and show $\mathcal{P}$.

There are no specific rules for proving and using *not* statements—which are called *negations.* Symbolically, the statement "not $\mathcal{P}$" is written $\neg\mathcal{P}$. Thus $\neg\mathcal{P}$ is true when $\mathcal{P}$ is false, and vice versa. Negations are handled by (1) proof by contradiction[13] and (2) logic propositions proved from axioms such as $\neg(\neg\mathcal{P})$ being equivalent to $\mathcal{P}$.

There remain, therefore, only the rules for proving and using *or* statements to complete the basis for the proof outlining procedure. Theorem 4 on the continuity of composites will illustrate the rule for using *or* statements. It is based on the following definitions:

Definition If f and g are functions, the *composite* function $f{\circ}g$ ("f circle g") is defined by

$$(f{\circ}g)(x) = f(g(x)).$$

The domain of $f{\circ}g$ consists of the numbers x in the domain of g for which $g(x)$ lies in the domain of f.

Definition A function f is *continuous* at (an interior point[14]) $x = c$ of its domain, if

$$\lim_{x\to c} f(x) = f(c).$$

[13] To prove a statement at the bottom of a gap by contradiction, assume (at the top of the gap) the negation of the statement, and then argue to a *contradiction*—by which we mean the negation of some statement already known or assumed to be true. In general, the proof outline method will have nothing to say about the steps needed to establish the contradiction, so it is generally best to exhaust the outline before resorting to a proof by contradiction.

[14] By considering only continuity at interior points, we avoid special cases involving one sided limits, without affecting the proof outline that we wish to illustrate.

Theorem 4 Continuity of Composites. If f is continuous at c, and g is continuous at $f(c)$, then $g \circ f$ is continuous at c.

Proof:

Assume f is continuous at c, and g is continuous at $f(c)$. Let D be the domain of $g \circ f$ and E be the domain of g. We will show $g \circ f$ is continuous at c.

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$g \circ f$ is continuous at c

The conclusion is, at the top level, an assertion of continuity. The definition of continuity gives us the next-to-last step:

Proof:

Assume f is continuous at c, and g is continuous at $f(c)$. Let D be the domain of $g \circ f$ and E be the domain of g. We will show $g \circ f$ is continuous at c.

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$\lim_{x \to c} g \circ f(x) = g \circ f(c)$

$g \circ f$ is continuous at c

The definition of limit provides the statement to be added to the bottom of the gap:

Proof:

Assume f is continuous at c, and g is continuous at $f(c)$. Let D be the domain of $g \circ f$ and E be the domain of g. We will show $g \circ f$ is continuous at c.

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For all $\epsilon > 0$, there exists $\delta > 0$ such that for all $x \in D$, if $0 < |x - c| < \delta$, then $|g \circ f(x) - g \circ f(c)| < \epsilon$.

$\lim_{x \to c} g \circ f(x) = g \circ f(c)$

$g \circ f$ is continuous at c

We now replace this *for-all* statement with the steps needed to prove it:

Proof:

Assume f is continuous at c, and g is continuous at $f(c)$. Let D be the domain of $g \circ f$ and E be the domain of g. We will show $g \circ f$ is continuous at c.

Let $\epsilon > 0$ be given.

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there exists $\delta > 0$ such that for all $x \in D$, if $0 < |x - c| < \delta$, then $|g \circ f(x) - g \circ f(c)| < \epsilon$.

$\lim_{x \to c} g \circ f(x) = g \circ f(c)$

$g \circ f$ is continuous at c

We now replace this *there-exists* statement with steps needed to prove it:

Proof:

Assume f is continuous at c, and g is continuous at $f(c)$. Let D be the domain of $g \circ f$ and E be the domain of g. We will show $g \circ f$ is continuous at c.

Let $\epsilon > 0$ be given.

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Let $\delta = \ldots$

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For all $x \in D$, if $0 < |x - c| < \delta$, then $|g \circ f(x) - g \circ f(c)| < \epsilon$.

$\lim_{x \to c} g \circ f(x) = g \circ f(c)$

$g \circ f$ is continuous at c

Proving the *for-all-if-then* statement:

Proof:

Assume f is continuous at c, and g is continuous at $f(c)$. Let D be the domain of $g \circ f$ and E be the domain of g. We will show $g \circ f$ is continuous at c.

Let $\epsilon > 0$ be given.

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Let $\delta = \ldots$

Assume $0 < |x - c| < \delta$ for an arbitrary $x \in D$.

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$|g \circ f(x) - g \circ f(c)| < \epsilon$.

$\lim_{x \to c} g \circ f(x) = g \circ f(c)$

$g \circ f$ is continuous at c

We can continue working up from the bottom, using the definition of composition:

Proof:

Assume f is continuous at c, and g is continuous at $f(c)$. Let D be the domain of $g \circ f$ and E be the domain of g. We will show $g \circ f$ is continuous at c.

Let $\epsilon > 0$ be given.

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Let $\delta = \ldots$

Assume $0 < |x - c| < \delta$ for an arbitrary $x \in D$.

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$|g(f(x)) - g(f(c))| < \epsilon$

$|g \circ f(x) - g \circ f(c)| < \epsilon$.

$\lim_{x \to c} g \circ f(x) = g \circ f(c)$

$g \circ f$ is continuous at c

The hypothesis that g is continuous at $f(c)$ states that $\lim_{y \to f(c)} g(y) = g(f(c)$ — which, by the definition of limit states that "for all $\epsilon' > 0$, there exists $\delta' > 0$ such that for all $y \in E$, if $0 < |y - f(c)| < \delta'$, then $|g(y) - g(f(c))| < \epsilon'$ ". Comparing the last inequality in this statement with the inequality at the bottom of the gap, we see that we need to apply the "for all $\epsilon' > 0, \ldots$" statement with $\epsilon' = \epsilon$. Thus we can add the following line:

Proof:

Assume f is continuous at c, and g is continuous at $f(c)$. Let D be the domain of $g \circ f$ and E be the domain of g. We will show $g \circ f$ is continuous at c.

Let $\epsilon > 0$ be given.

By the continuity of g and the definition of limit, there exists $\delta' > 0$ such that for all $y \in E$, if $0 < |y - f(c)| < \delta'$, then $|g(y) - g(f(c))| < \epsilon$.

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Let $\delta = \ldots$

Assume $0 < |x - c| < \delta$ for an arbitrary $x \in D$.

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$|g(f(x)) - g(f(c))| < \epsilon$

$|g \circ f(x) - g \circ f(c)| < \epsilon$.

$\lim_{x \to c} g \circ f(x) = g \circ f(c)$

$g \circ f$ is continuous at c

Its also clear that we need to apply the *for-all* statement "for all $y \in E, \ldots$" in this added line. We apply it by taking $y = f(x)$—to conclude "if $0 < |f(x) - f(c)| < \delta'$, then $|g(f(x)) - g(f(c))| < \epsilon$". This needs to be added in the gap (after x is defined in the proof):

Proof:

Assume f is continuous at c, and g is continuous at $f(c)$. Let D be the domain of $g \circ f$ and E be the domain of g. We will show $g \circ f$ is continuous at c.

Let $\epsilon > 0$ be given.

By the continuity of g and the definition of limit, there exists $\delta' > 0$ such that for all $y \in E$, if $0 < |y - f(c)| < \delta'$, then $|g(y) - g(f(c))| < \epsilon$.

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Let $\delta = \ldots$

Assume $0 < |x - c| < \delta$ for an arbitrary $x \in D$.

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If $0 < |f(x) - f(c)| < \delta'$, then $|g(f(x)) - g(f(c))| < \epsilon$

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$|g(f(x)) - g(f(c))| < \epsilon$

$|g \circ f(x) - g \circ f(c)| < \epsilon$.

$\lim\limits_{x \to c} g \circ f(x) = g \circ f(c)$

$g \circ f$ is continuous at c

By the rule for using *if-then* statements, we need to show $0 < |f(x) - f(c)| < \delta'$, in order to get the result we're after. We can use the continuity of f to get the right hand inequality: by continuity, $\lim\limits_{x \to c} f(x) = f(c)$, and by the definition of limit: for all $\epsilon'' > 0$, there exists $\delta'' > 0$ such that for all $x \in D$, if $0 < |x - c| < \delta''$, then $|f(x) - f(c)| < \epsilon''$. We need to apply the statement with $\epsilon'' = \delta'$. This gives us a δ'', which is just the δ we need—so we call it δ in the first place:

Proof:

Assume f is continuous at c, and g is continuous at $f(c)$. Let D be the domain of $g \circ f$ and E be the domain of g. We will show $g \circ f$ is continuous at c.

Let $\epsilon > 0$ be given.

By the continuity of g and the definition of limit, there exists $\delta' > 0$ such that for all $y \in E$, if $0 < |y - f(c)| < \delta'$, then $|g(y) - g(f(c))| < \epsilon$.

By the continuity of f and the definition of limit, there exists $\delta > 0$ such that for all $x \in D$, if $0 < |x - c| < \delta$, then $|f(x) - f(c)| < \delta'$.

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Let $\delta = \ldots$

Assume $0 < |x - c| < \delta$ for an arbitrary $x \in D$.

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If $0 < |f(x) - f(c)| < \delta'$, then $|g(f(x)) - g(f(c))| < \epsilon$

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$|g(f(x)) - g(f(c))| < \epsilon$

$|g \circ f(x) - g \circ f(c)| < \epsilon$.

$\lim_{x \to c} g \circ f(x) = g \circ f(c)$

$g \circ f$ is continuous at c

The statement "let $\delta = \ldots$" that we roughed in is not necessary. The correct δ has been defined by the previous line. We'll also use t as the local variable in this line, in order to avoid the appearance of talking about x, before it has been defined. The proof now looks like this:

Proof:

Assume f is continuous at c, and g is continuous at $f(c)$. Let D be the domain of $g \circ f$ and E be the domain of g. We will show $g \circ f$ is continuous at c.

Let $\epsilon > 0$ be given.

By the continuity of g and the definition of limit, there exists $\delta' > 0$ such that for all $y \in E$, if $0 < |y - f(c)| < \delta'$, then $|g(y) - g(f(c))| < \epsilon$.

(*) By the continuity of f and the definition of limit, there exists $\delta > 0$ such that for all $t \in D$, if $0 < |t - c| < \delta$, then $|f(t) - f(c)| < \delta'$.

(**) Assume $0 < |x - c| < \delta$ for an arbitrary $x \in D$.

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If $0 < |f(x) - f(c)| < \delta'$, then $|g(f(x)) - g(f(c))| < \epsilon$

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$|g(f(x)) - g(f(c))| < \epsilon$

$|g \circ f(x) - g \circ f(c)| < \epsilon$.

$\lim_{x \to c} g \circ f(x) = g \circ f(c)$

$g \circ f$ is continuous at c

Note that δ has been defined in the line (*), so that we can use it in the next line (**). The symbol x, however, is defined in the line (**), so that we can't use it (as a global variable) previously. The definition of δ must come first. Then we apply the *for-all-if-then* statement "for all $t \in D, \ldots$" of line (*) to the x of line (**) to conclude $|f(x) - f(c)| < \delta'$. In order to give the reader an idea of where $|f(x) - f(c)| < \delta'$ comes from, we introduce it by the phrase "from the two previous lines we get".

Proof:

Assume f is continuous at c, and g is continuous at $f(c)$. Let D be the domain of $g \circ f$ and E be the domain of g. We will show $g \circ f$ is continuous at c.

Let $\epsilon > 0$ be given.

(*) By the continuity of g and the definition of limit, there exists $\delta' > 0$ such that for all $y \in E$, if $0 < |y - f(c)| < \delta'$, then $|g(y) - g(f(c))| < \epsilon$.

By the continuity of f and the definition of limit, there exists $\delta > 0$ such that for all $t \in D$, if $0 < |t - c| < \delta$, then $|f(t) - f(c)| < \delta'$.

Assume $0 < |x - c| < \delta$ for an arbitrary $x \in D$.

From the two previous lines, we get $|f(x) - f(c)| < \delta'$.

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If $0 < |f(x) - f(c)| < \delta'$, then $|g(f(x)) - g(f(c))| < \epsilon$

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$|g(f(x)) - g(f(c))| < \epsilon$

$|g \circ f(x) - g \circ f(c)| < \epsilon$.

$\lim_{x \to c} g \circ f(x) = g \circ f(c)$

$g \circ f$ is continuous at c

Note that the statement "if $0 < |f(x) - f(c)| < \delta'$, then $|g(f(x)) - g(f(c))| < \epsilon$" in the gap came from the line now marked (*). The conclusion of this implication, $|g(f(x)) - g(f(c))| < \epsilon$, is just the thing we need. In order to use the implication, however, we need $0 < |f(x) - f(c)| < \delta'$ to hold, but we only have $|f(x) - f(c)| < \delta'$ from the proven lines. We also need $0 < |f(x) - f(c)|$, in order to use the implication—and this isn't necessarily true. It certainly could be the case that $f(x) = f(c)$. The way out of this dilemma is to break the proof into cases. The introduction of cases in a proof is formally justified by a rule for using *or* statements.

Inference Rule Using *or* statements. In order to show that a statement $\mathcal{R}$ follows from a statement "$\mathcal{P}$ or $\mathcal{Q}$", assume as a first case that $\mathcal{P}$ is true, and show that $\mathcal{R}$ follows from $\mathcal{P}$, then assume as a second case that $\mathcal{Q}$ is true, and show that $\mathcal{R}$ follows from $\mathcal{Q}$.

The more general rule involves a statement $\mathcal{P}_1$ or $\mathcal{P}_2$ or $\mathcal{P}_3$ or $\ldots$ or $\mathcal{P}_n$ with n constituent statements, and therefore n cases in a proof. In order to show that $\mathcal{R}$ follows from such a statement, show that $\mathcal{R}$ follows in all cases that don't lead to a contradiction.

In order to use an *or* statement in our proof, we insert the line "$f(x) = f(c)$ or $f(x) \neq f(c)$"—which holds by the basic logic assumed in mathematics. This leads to 2 cases in the proof. We need to show that the result we're after (the statement $|g(f(x)) - g(f(c))| < \epsilon$ at the bottom of the gap) holds in both cases.

Proof:

Assume f is continuous at c, and g is continuous at $f(c)$. Let D be the domain of $g \circ f$ and E be the domain of g. We will show $g \circ f$ is continuous at c.

Let $\epsilon > 0$ be given.

(*) By the continuity of g and the definition of limit, there exists $\delta' > 0$ such that for all $y \in E$, if $0 < |y - f(c)| < \delta'$, then $|g(y) - g(f(c))| < \epsilon$.

By the continuity of f and the definition of limit, there exists $\delta > 0$ such that for all $t \in D$, if $0 < |t - c| < \delta$, then $|f(t) - f(c)| < \delta'$.

Assume $0 < |x - c| < \delta$ for an arbitrary $x \in D$.

(**) From the two previous lines, we get $|f(x) - f(c)| < \delta'$.

$f(x) = f(c)$ or $f(x) \neq f(c)$

Assume as Case 1 that $f(x) = f(c)$. Then $g(f(x)) = g(f(c))$, so that $|g(f(x)) - g(f(c))| < \epsilon$

Assume as Case 2 that $f(x) \neq f(c)$. Then $0 < |f(x) - f(c)|$. From this and (**), we get $0 < |f(x) - f(c)| < \delta'$, but

from (*) we get if $0 < |f(x) - f(c)| < \delta'$, then $|g(f(x)) - g(f(c))| < \epsilon$.

Thus $|g(f(x)) - g(f(c))| < \epsilon$ holds in Case 2.

Therefore $|g(f(x)) - g(f(c))| < \epsilon$ holds in any event.

Therefore $|g \circ f(x) - g \circ f(c)| < \epsilon$, by the definition of composition.

Thus $\lim_{x \to c} g \circ f(x) = g \circ f(c)$, by the definition of limit, so

$g \circ f$ is continuous at c.

□

For the sake of completeness, we mention the rule for *proving or* statements.

Inference Rule Proving *or* statements. In order to prove the statement "$\mathcal{P}$ or $\mathcal{Q}$", either assume that $\mathcal{P}$ is false and show $\mathcal{Q}$, or assume that $\mathcal{Q}$ is false, and show $\mathcal{P}$.

EXERCISES

5. Let f be defined as follows:

$$f(x) = \begin{cases} x + 1, & \text{if } x \leq 1 \\ 4 - 2x, & \text{if } x > 1 \end{cases}$$

Prove that $\lim_{x \to 1} f(x) = 2$.

6. Show that the function $f(x) = x^2$ is continuous at $x = 2$.

Derivatives

Probably the most widely used theorem in calculus asserts that the derivative of a sum is the sum of the derivatives: in symbols, $(f+g)' = f' + g'$. This theorem allows us to differentiate polynomials and other expressions term by term. We need the following definitions:

Definition For a function $f\colon D \to \mathbb{R}$ and $x \in D$, f is said to be *differentiable at* x if $\lim\limits_{h\to 0} \frac{f(x+h)-f(x)}{h}$ exists. If f is differentiable at all $x \in D$, then f is said to be *differentiable on* D.

Definition If $f\colon D \to \mathbb{R}$ is differentiable on D, the function $f'\colon D \to \mathbb{R}$ defined by $f'(x) = \lim\limits_{h\to 0} \frac{f(x+h)-f(x)}{h}$ is called the derivative of f.

Theorem 5 The sum rule: If f and g are differentiable on D, then their sum $f+g$ is differentiable on D and

$$(f+g)' = f' + g'.$$

Proof:

Assume that f and g are differentiable on D. We will show (1) $f+g$ is differentiable on D, and (2) $(f+g)' = f' + g'$.

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(1) $f+g$ is differentiable on D

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(2) $(f+g)' = f' + g'$

By the definition of "differentiable on D", part (1) of the conclusion is that for all $x \in D$, $\lim\limits_{h\to 0} \frac{(f+g)(x+h)-(f+g)(x)}{h}$ exists. Therefore we add the following steps to the gap above part (1):

Proof:

Assume that f and g are differentiable on D. We will show (1) $f+g$ is differentiable on D, and (2) $(f+g)' = f' + g'$.
Let $x \in D$ be arbitrary.
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.
.
$\lim_{h\to 0} \frac{(f+g)(x+h)-(f+g)(x)}{h}$ exists.
(1) $f+g$ is differentiable on D.
.
.
.
(2) $(f+g)' = f' + g'$

The statement in the proof asserting the existence of the limit could be formalized as "there exists $L \in \mathbb{R}$, such that $L = \lim_{h\to 0} \frac{(f+g)(x+h)-(f+g)(x)}{h}$. To prove that such a number L exists, we define it. By hypotheses, $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ and $\lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$ both exist, so we define L to be the sum of these:

Proof:

Assume that f and g are differentiable on D. We will show (1) $f+g$ is differentiable on D, and (2) $(f+g)' = f' + g'$.
Let $x \in D$ be arbitrary.
By hypotheses, $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ and $\lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$ both exist.
Let $L = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$.
.
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.
$\lim_{h\to 0} \frac{(f+g)(x+h)-(f+g)(x)}{h}$ exists.
(1) $f+g$ is differentiable on D.
.
.
.
(2) $(f+g)' = f' + g'$

Proof:

Assume that f and g are differentiable on D. We will show (1) $f+g$ is differentiable on D, and (2) $(f+g)' = f' + g'$.
Let $x \in D$ be arbitrary.
By hypotheses, $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ and $\lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$ both exist.
Let $L = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$.
Then $L = \lim_{h\to 0} \left[\frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h} \right]$ (by Theorem 1)
$= \lim_{h\to 0} \frac{f(x+h)+g(x+h)-f(x)-g(x)}{h}$ (adding the fractions)
$= \lim_{h\to 0} \frac{(f+g)(x+h)-(f+g)(x)}{h}$ (by definition of the function $f+g$).
Therefore $\lim_{h\to 0} \frac{(f+g)(x+h)-(f+g)(x)}{h}$ exists.
(1) $f+g$ is differentiable on D.
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.
.
(2) $(f+g)' = f' + g'$

The second part of the conclusion, at the top level, is an assertion that two functions are equal. By definition, this means that they are have equal values at every point of their domain D. This definition supplies us with the next-to-the-last line:

Proof:

Assume that f and g are differentiable on D. We will show (1) $f+g$ is differentiable on D, and (2) $(f+g)' = f' + g'$.
Let $x \in D$ be arbitrary.
By hypotheses, $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ and $\lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$ both exist.
Let $L = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$.
Then $L = \lim_{h\to 0} \left[\frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h} \right]$ (by Theorem 1)
$= \lim_{h\to 0} \frac{f(x+h)+g(x+h)-f(x)-g(x)}{h}$ (adding the fractions)
$= \lim_{h\to 0} \frac{(f+g)(x+h)-(f+g)(x)}{h}$ (by definition of the function $f+g$).
Therefore $\lim_{h\to 0} \frac{(f+g)(x+h)-(f+g)(x)}{h}$ exists.
(1) $f+g$ is differentiable on D.
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.
.
For all $x \in D$, $(f+g)'(x) = f'(x) + g'(x)$
Therefore (2) $(f+g)' = f' + g'$ by definition of equal functions.

The rule for proving *for-all* statements leads us to replace the next-to-the-last line with the lines needed to prove it:

Proof:

Assume that f and g are differentiable on D. We will show (1) $f+g$ is differentiable on D, and (2) $(f+g)' = f' + g'$.
Let $x \in D$ be arbitrary.
By hypotheses, $\lim\limits_{h\to 0} \frac{f(x+h)-f(x)}{h}$ and $\lim\limits_{h\to 0} \frac{g(x+h)-g(x)}{h}$ both exist.
Let $L = \lim\limits_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim\limits_{h\to 0} \frac{g(x+h)-g(x)}{h}$.
Then $L = \lim\limits_{h\to 0}\left[\frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h}\right]$ (by Theorem 1)
$= \lim\limits_{h\to 0} \frac{f(x+h)+g(x+h)-f(x)-g(x)}{h}$ (adding the fractions)
$= \lim\limits_{h\to 0} \frac{(f+g)(x+h)-(f+g)(x)}{h}$ (by definition of the function $f+g$).
Therefore $\lim\limits_{h\to 0} \frac{(f+g)(x+h)-(f+g)(x)}{h}$ exists.
(1) $f+g$ is differentiable on D.
Let $x \in D$ be arbitrary.
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.
.
$(f+g)'(x) = f'(x) + g'(x)$
(2) $(f+g)' = f' + g'$ by definition of equal functions.

We work up from the bottom of the gap, using the definition of the functions f', g', and $(f+g)'$:

Proof:

Assume that f and g are differentiable on D. We will show (1) $f+g$ is differentiable on D, and (2) $(f+g)' = f' + g'$.
Let $x \in D$ be arbitrary.
By hypotheses, $\lim\limits_{h\to 0} \frac{f(x+h)-f(x)}{h}$ and $\lim\limits_{h\to 0} \frac{g(x+h)-g(x)}{h}$ both exist.
Let $L = \lim\limits_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim\limits_{h\to 0} \frac{g(x+h)-g(x)}{h}$.
Then $L = \lim\limits_{h\to 0}\left[\frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h}\right]$ (by Theorem 1)
$= \lim\limits_{h\to 0} \frac{f(x+h)+g(x+h)-f(x)-g(x)}{h}$ (adding the fractions)
$= \lim\limits_{h\to 0} \frac{(f+g)(x+h)-(f+g)(x)}{h}$ (by definition of the function $f+g$).
Therefore $\lim\limits_{h\to 0} \frac{(f+g)(x+h)-(f+g)(x)}{h}$ exists.
(1) $f+g$ is differentiable on D.
Let $x \in D$ be arbitrary.
.
.
.
$\lim\limits_{h\to 0} \frac{(f+g)(x+h)-(f+g)(x)}{h} = \lim\limits_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim\limits_{h\to 0} \frac{g(x+h)-g(x)}{h}$
$(f+g)'(x) = f'(x) + g'(x)$
(2) $(f+g)' = f' + g'$ by definition of equal functions.

The statement at the bottom of the gap has already been shown to be true for an arbitrary $x \in D$, in the first part of the proof. This gives a complete proof:

Proof:

Assume that f and g are differentiable on D. We will show (1) $f+g$ is differentiable on D, and (2) $(f+g)' = f' + g'$.
Let $x \in D$ be arbitrary.
By hypotheses, $\lim\limits_{h\to 0} \frac{f(x+h)-f(x)}{h}$ and $\lim\limits_{h\to 0} \frac{g(x+h)-g(x)}{h}$ both exist.
Let $L = \lim\limits_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim\limits_{h\to 0} \frac{g(x+h)-g(x)}{h}$.
Then $L = \lim\limits_{h\to 0} \left[\frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h} \right]$ (by Theorem 1)
$= \lim\limits_{h\to 0} \frac{f(x+h)+g(x+h)-f(x)-g(x)}{h}$ (adding the fractions)
$= \lim\limits_{h\to 0} \frac{(f+g)(x+h)-(f+g)(x)}{h}$ (by definition of the function $f+g$).
Therefore $\lim\limits_{h\to 0} \frac{(f+g)(x+h)-(f+g)(x)}{h}$ exists—proving (1).

Again, let $x \in D$ be arbitrary. Then as we have seen,
$\lim\limits_{h\to 0} \frac{(f+g)(x+h)-(f+g)(x)}{h} = \lim\limits_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim\limits_{h\to 0} \frac{g(x+h)-g(x)}{h}$.
Therefore $(f+g)'(x) = f'(x) + g'(x)$ by definition of the functions f', g', and $(f+g)'$:
This shows (2).

□

Summary

The proof outlining method depends on the fact that mathematics can be formalized by only a few different types of statements, and each of these types has a very routine way of being proved or used. It depends on the fact that everything in mathematics is just what its definition says it is, so that a proof that something has a property must be, ultimately, a demonstration that the property follows logically from the definition.

The proof outlining method works by filling in gaps in a proof with steps given by the explicit inference rules. This results in complete proofs of straightforward consequences of definitions. More frequently, a proof outline with a gap to be bridged remains. In this case, the outlining method identifies just what is needed to bridge the gap.

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