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Logical Deduction ... is the one and only true powerhouse of mathematical thinking.

Jean Dieudonne

Conjecturing and demonstrating the logical validity of conjectures are the essence of the creative act of doing mathematics.

NCTM *Standards*

Introduction

Deductive versus Descriptive Mathematics

Mathematics has two fundamental aspects: (1) discovery/logical deduction and (2) description/computation. Discovery/deductive mathematics asks the questions:

1. What is true about this thing being studied?
2. How do we know it is true?

On the other hand, descriptive/computational mathematics asks questions of the type:

3. What is the particular number, function, and so on, that satisfies ... ?
4. How can we find the number, function, and so on?

In descriptive/computational mathematics, some pictorial, physical, or business situation is described mathematically, and then computational techniques are applied to the mathematical description, in order to find values of interest. The foregoing is frequently called “problem solving”. Examples of the third question such as “How many feet of fence will be needed by a farmer to enclose ...” are familiar. The fourth type of question is answered by techniques such as solving equations, multiplying whole numbers, finding antiderivatives, substituting in formulas, and so on. The first two questions, however, are unfamiliar to most. The teaching of computational techniques continues to be the overwhelming focus of mathematics education. For most people, the techniques, and their application to real world or business problems, *are* mathematics. Mathematics is understood only in its descriptive role in providing a language for scientific, technical, and business areas.

Mathematics, however, is really a deductive science. Mathematical knowledge comes from people looking at examples, and getting an idea of what may be true in general. Their idea is put down formally as a statement—a conjecture. The statement is then shown to be a logical consequence of what we already know. The way this is done is by logical deduction. The mathematician Jean Dieudonne has called logical deduction “the one and only true powerhouse of mathematical thinking”[1]. Finding proofs for conjectures is also called “problem solving”. The “Problems” sections of several mathematics journals for students and teachers involve primarily problems of this type.

The deductive and descriptive aspects of mathematics are complementary—not antagonistic—they motivate and enrich each other. The relation between the two aspects has been a source of wonder to thoughtful people[2].

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[1] J. Dieudonne, *Linear Algebra and Geometry*, Hermann, Paris, 1969, page 14.

[2] John Polkinghorne in his *The Way the World Is* (Wm B. Eerdmans, Grand Rapids, MI, 1984, page 9) states, “Again and again in physical science we find that it is the abstract structures of pure mathematics which provide the clue to understanding the world. It is a recognized *technique* in fundamental physics to seek theories which have an elegant and economical (you can say beautiful) mathematical form, in the expectation that they will prove the ones realized in nature. General relativity, the modern theory of gravitation, was invented by Einstein in just such a way. Now mathematics is the free creation of the human mind, and it is surely a surprising and significant thing that a discipline apparently so unearthed should provide the key with which to turn the lock of the world.”

Mathematics Education

In 1941, Richard Courant and Herbert Robbins published their book, *What is Mathematics?—An Elementary Approach to Ideas and Methods*[3]. In the preface (to the first edition) we read,

> Today the traditional place of mathematics in education is in grave danger. Unfortunately, professional representatives of mathematics share in the responsibility. The teaching of mathematics has sometimes degenerated into empty drill in problem solving, which may develop formal ability but does not lead to real understanding or to greater intellectual independence.
>
> It is possible to proceed on a straight road from the very elements to vantage points from which the substance and driving forces of modern mathematics can be surveyed. The present book is an attempt in this direction.[4]

From the preface to the second, third and fourth editions (1943, '45 '47), we read:

> Now more than ever there exists the danger of frustration and disillusionment unless students and teachers try to look beyond mathematical formalism and manipulation and to grasp the real essence of mathematics. This book was written for such students and teachers, . . .[5]

Some 20 years later, a trend in mathematics education called the “new math” would incorporate many of the “very elements” in Courant and Robbins' book: for example, the representation of integers in terms of powers of a base—the standard base 10 and other bases, computations in systems other than the decimal, the foundational role played by the commutative, associative, and distributive axioms for the integers, and the introduction of the language and ideas of sets: “The concept of a class or set of objects is one of the most fundamental in mathematics.”[6]

One pervading theme in Courant and Robbins that never was incorporated into the new math is the centrality of proof to mathematics. The new math used the language of deductive mathematics to shed light on and do descriptive mathematics (sometimes awkwardly). Merely shedding light on “mathematical formalism and manipulation” and failing to shed much light on “problem solving”, the curriculum changes introduced by the new math have largely faded from the school curriculum. Although fading from the school curriculum, elements of the new math curriculum have been maintained in college courses for prospective elementary school teachers—in particular, the “axioms of arithmetic” as a basis for operations and computations in the systems of natural numbers, integers, and rational numbers. In this text, attention is paid to these number systems, and they are included within the framework of deductive mathematics—whereas in *Introduction to Proof in Abstract Mathematics*, the computations of algebra are accepted, where needed, even in a formal proof. In this text, the logical foundation for these computations is made explicit.

Standards for Curriculum Change

The Mathematical Association of America (MAA) publication *A Call For Change* addresses the needs of prospective teachers. Their recommendations summarize publications of the National Research Council (NRC) and the Standards of the National Council of Teachers of Mathematics (NCTM):

> There is an overwhelming consensus that students of the 1990's and beyond will develop “mathematical power” only if they are actively involved in **doing**[7] mathematics at every grade level . “Mathematical power” denotes a person's abilities to explore,

[3] Oxford University Press, London, New York, Toronto
[4] page v
[5] page vii
[6] page 108
[7] Emphasis in original.

> conjecture, and reason logically, as well as use a variety of mathematical methods effectively to solve problems.
>
> Such substantive changes in school mathematics will require corresponding changes in the preparation of teachers.[8]

The summary breaks things into 2 fundamental aspects: (1) explore, conjecture, and reason logically, and (2) use a variety of mathematical methods effectively to solve problems.[9] For mathematics to be properly understood, the essence of what it is, as a deductive science itself *and* as a language for other areas, should be seen at all levels. An understanding of the scientific method is not thought to be appropriate only for a few research scientists. The rudiments and purposes of the scientific method can and should be taught in the most elementary science courses. The same should be true for mathematics. Just as science needs to be taught as more than technology, mathematics needs to be taught as more than techniques. This need has been addressed in the calls for reform.

Standard 1 in *A Call for Change*, "Learning Mathematical Ideas", applicable to teachers of mathematics at *all* grade levels, includes:

> "Exercise mathematical reasoning through recognizing patterns, making and refining conjectures and definitions, and constructing logical arguments, both formal and heuristic, to justify results."

The NCTM *Curriculum and Evaluation Standards for School Mathematics* (section for grades 5 through 8) states:

> Conjecturing and demonstrating the logical validity of conjectures are the essence of the creative act of doing mathematics.[10]

The *Standards* say a lot of things, but there is only one thing that they have called the "essence" of doing mathematics. The context for this sort of activity—what the NCTM evidently had in mind when making the statement—is the geometry taught in the schools. According to current thinking, students pass through stages in their geometric thinking. The ability to appreciate proof—especially rigorous proof—occurs at a late stage, intuitive perceptions occur at earlier stages, and it is not possible to get to the later stages without a lengthy maturing process that takes one through the earlier stages. What this means for discovery/deductive mathematics, is that students will be making conjectures based on pattern recognition, and not for some years be able to demonstrate the logical validity of their conjectures.

This text presents a system designed to enable students to find and construct their own logical arguments. The system is first applied to elementary ideas about sets and subsets and the set operations of union, intersection, and difference—which are now generally introduced prior to high school. These set operations and relations so closely follow the logic used in elementary mathematical arguments, that students using the system are naturally prepared to prove any (true) conjectures they might discover about them. It is an easy entry into the world of discovery/ deductive mathematics. It enables students to verify the validity of their own conjectures—as the conjectures are being made.

A Bottom-Up Approach

The system is based on a bottom-up approach. Certain things are best learned from the bottom up: programming in a specific programming language, for example, or learning how to play chess. In the bottom-up learning, there ought to be no doubt of what constitutes a valid chess move on a valid chess board. Other things, such as speaking in one's own native language, are learned from the top down. As we learn to speak, grammar (which would be analogous to the rules of the game

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[8] The Mathematical Association of America, *A Call for Change*, report of the Committee On The Mathematical Education Of Teachers, J. R. C. Leitzel, Ed., 1991, preface.
[9] What is likely meant by "solve problems" at this level is to apply mathematics to physical, pictorial, or business situations. At more advanced levels, problem solving predominantly means supplying proof for conjectures.
[10] NCTM, 1989, page 81.

for chess) is not even part of our consciousness. Grammatical rules are followed only because they are used implicitly by those that we imitate. If the people around us use poor grammar, we nevertheless learn to feel it is “right”—and we speak the same way.

The system in this text is based on a number of formal inference rules that model what a mathematician would do naturally to prove certain sorts of statements. The rules make explicit the logic used implicitly by mathematicians.[11] After experience is gained, the explicit use of the formal rules is replaced by implicit reference. Thus, in our bottom-up approach, the explicit precedes the implicit. The initial, formal step-by-step format (which allows for the explicit reference to the rules) is replaced by a narrative format—where only critical things need to be mentioned Thus the student is lead up to the sort of narrative proofs traditionally found in text books. At every stage in the process, the student is always aware of what is and what is not a proof—and has specific guidance in the form of a “step discovery procedure” that leads to a proof outline.

The system has been used extensively in courses for prospective elementary-school teachers. Diligent students learn the material.[12] Sections 1 through 15 of the text are devoted to producing basic skill with logical reasoning. Section 16 presents the first of the student investigations. In this first investigation, students have discovered in the past a number of important relationships between the set operations of intersection, union, and difference—and have been able to supply their own completely rigorous, well-written proofs of their conjectures.

A Course Based on the Text

A Call For Change recommends 3 college courses in mathematics for prospective K–4 teachers, and 15 hours for prospective 5–8 teachers. A course based on this text would probably be best placed after 2 courses based on more traditional material. Because such a course differs from the traditional approach, however, some bright students have benefited from having it as their first college course in mathematics.[13] Appendices 1 through 3 present material that can be used for a review of computationally oriented mathematics probably already in the students' experience. Appendix 5 contains a basic syllabus, and Appendix 6, a syllabus for a more advanced class.

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[11] Although the rules resemble those of formal logic, they were developed solely to help students struggling with proof—without any input from formal logic.

[12] A correlation of 0.7 has been found between the scores of students on final exams over the text material and their ranking in their high-school class. Lower correlations, 0.5 and 0.2 respectively, have been found with students' math and verbal SAT scores.

[13] For example, one student wrote, “This course has made interesting a subject I used to hate.” Another wrote, “Doing deductive mathematics is more interesting than doing computational mathematics.”